GLOBAL OPTIMIZATION WITH BRANCH-AND-REDUCE

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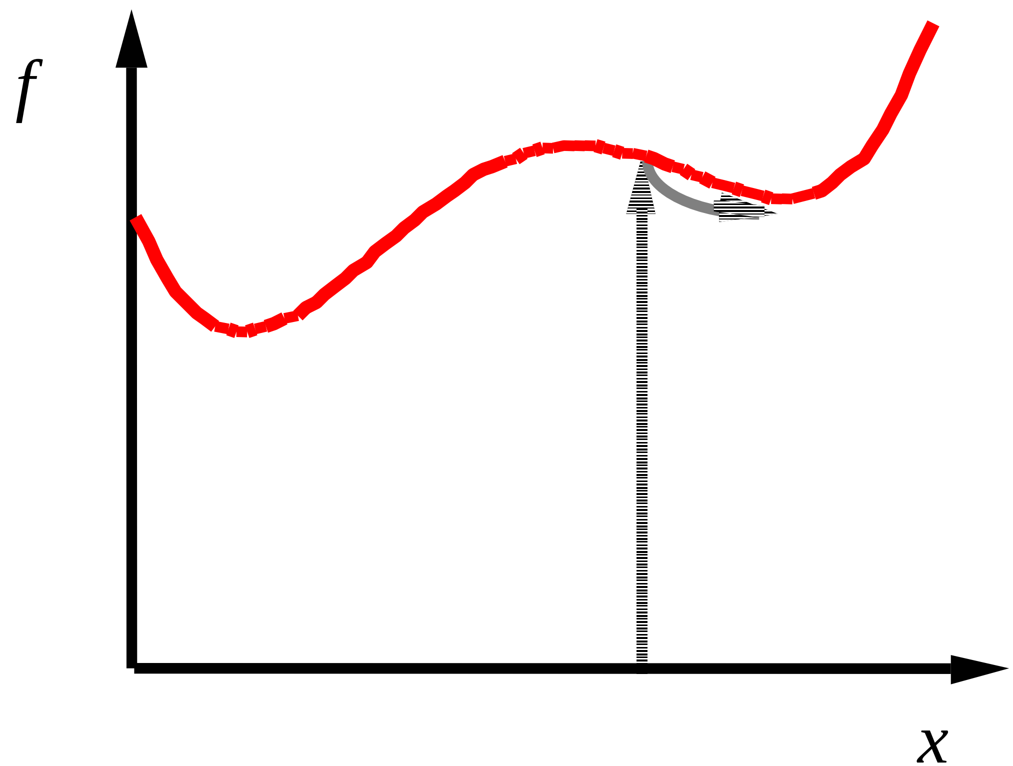
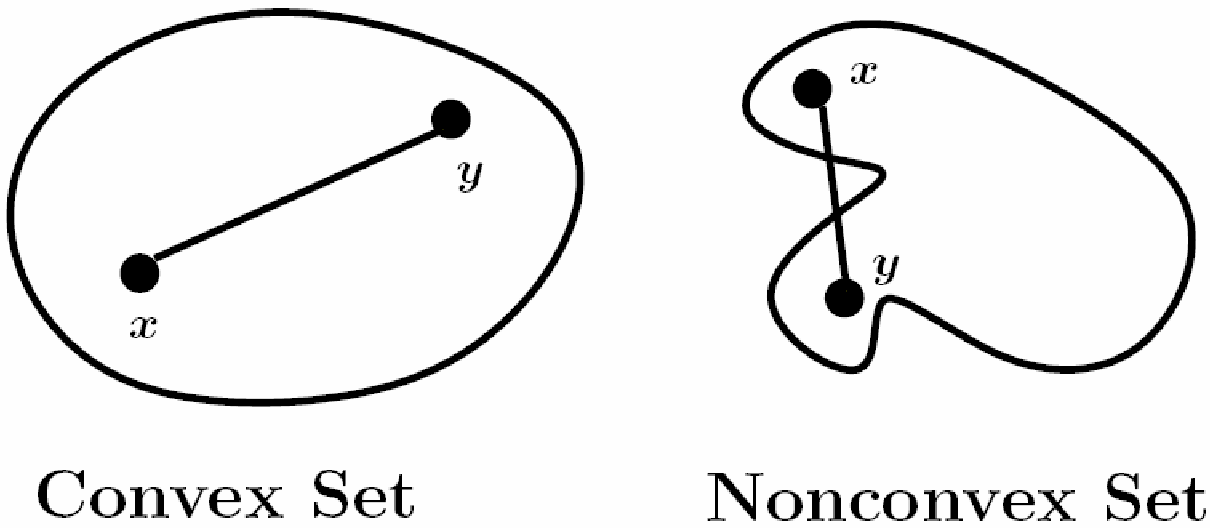
THE MULTIPLE-MINIMA DIFFICULTY

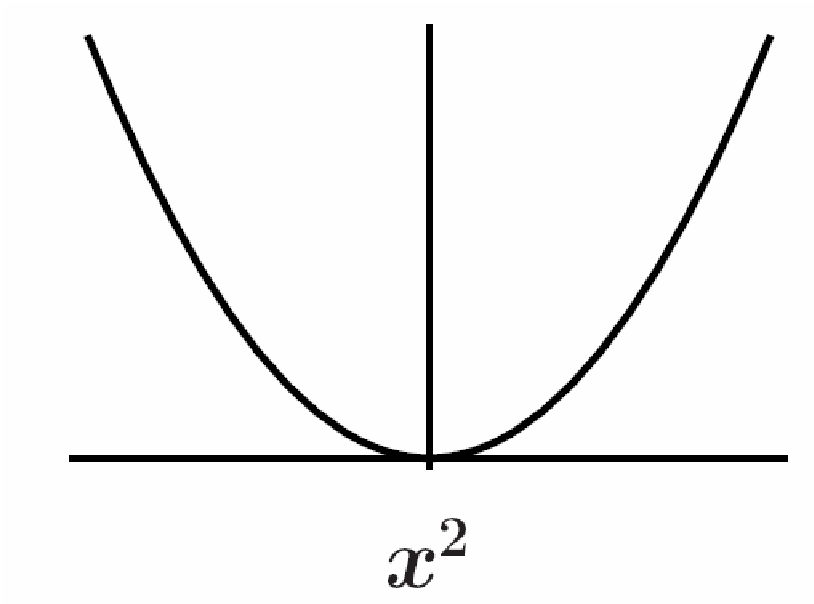


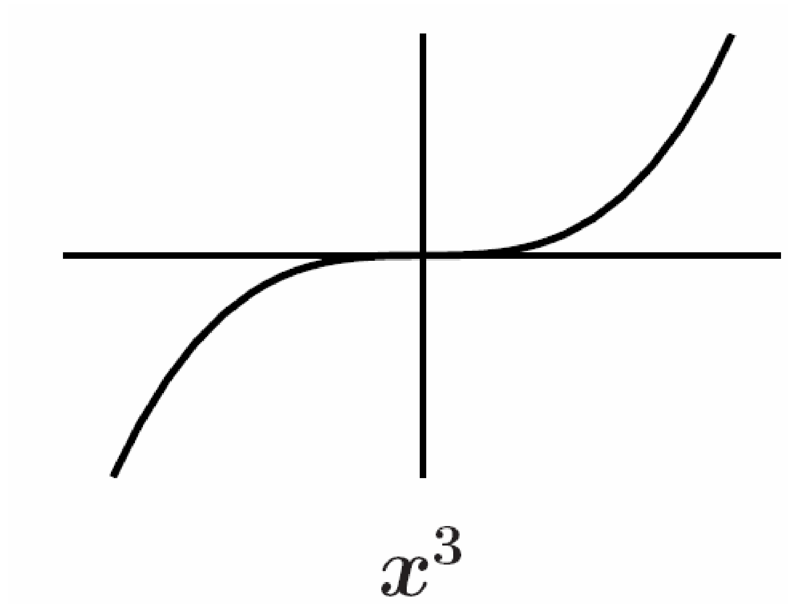
- **Classical optimality conditions are necessary but not sufficient**
- **Classical optimization provides the local minimum “closest” to the starting point used**

CONVEX AND NONCONVEX SETS



COMMON FUNCTIONS IN MODELING





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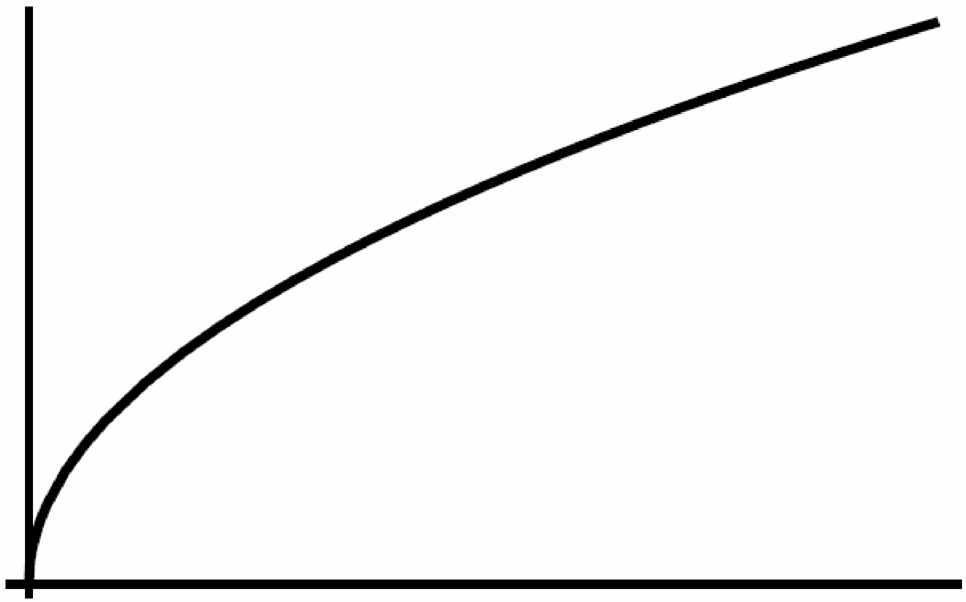
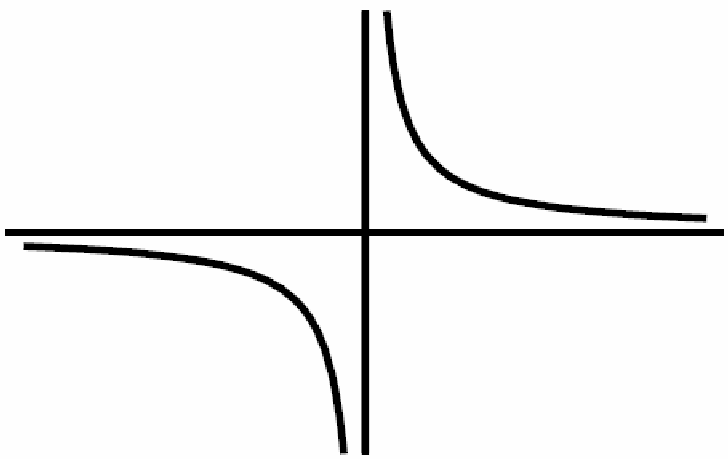
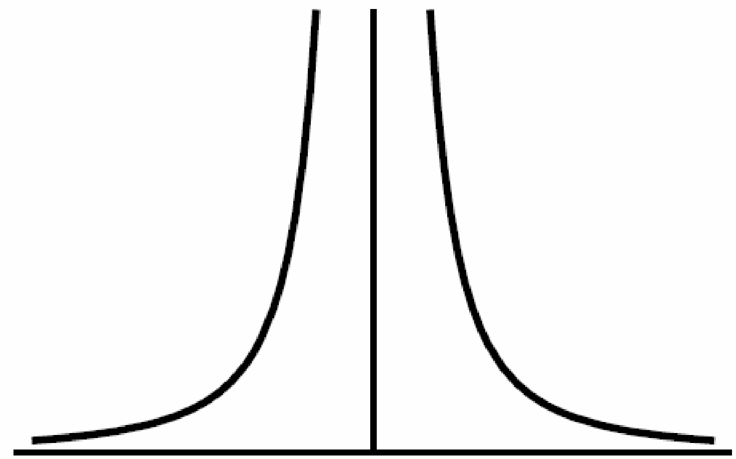


$$\sqrt{x}$$

COMMON FUNCTIONS IN MODELING

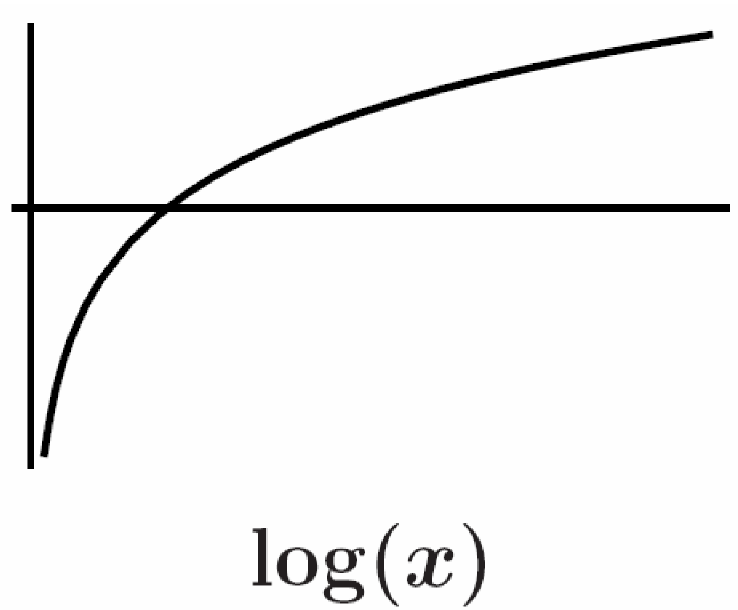


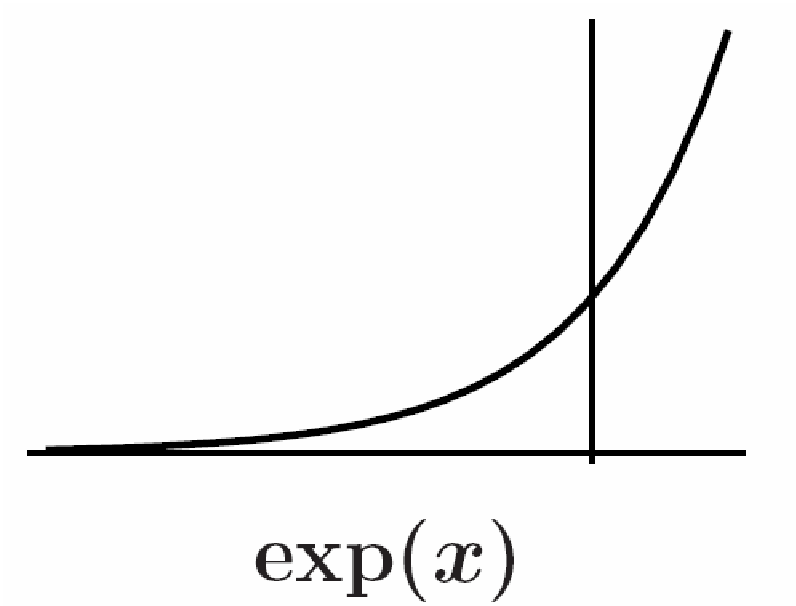
$$\frac{1}{x}$$



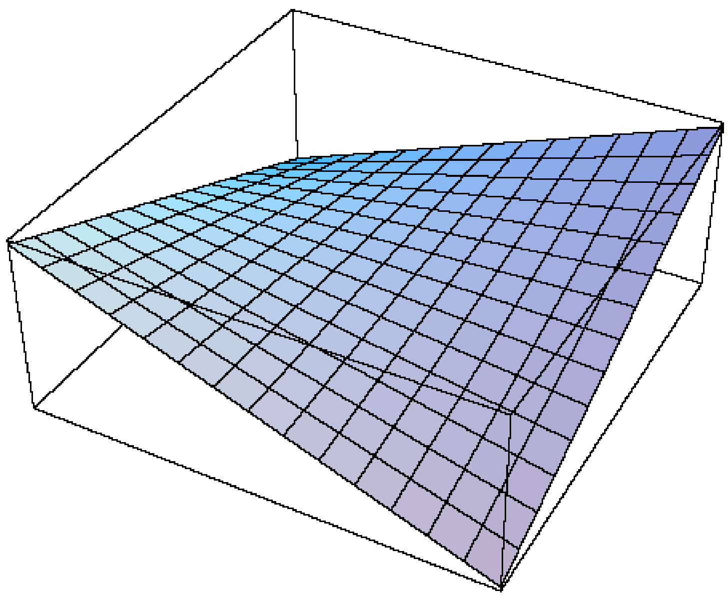
$$\frac{1}{x^2}$$

COMMON FUNCTIONS IN MODELING

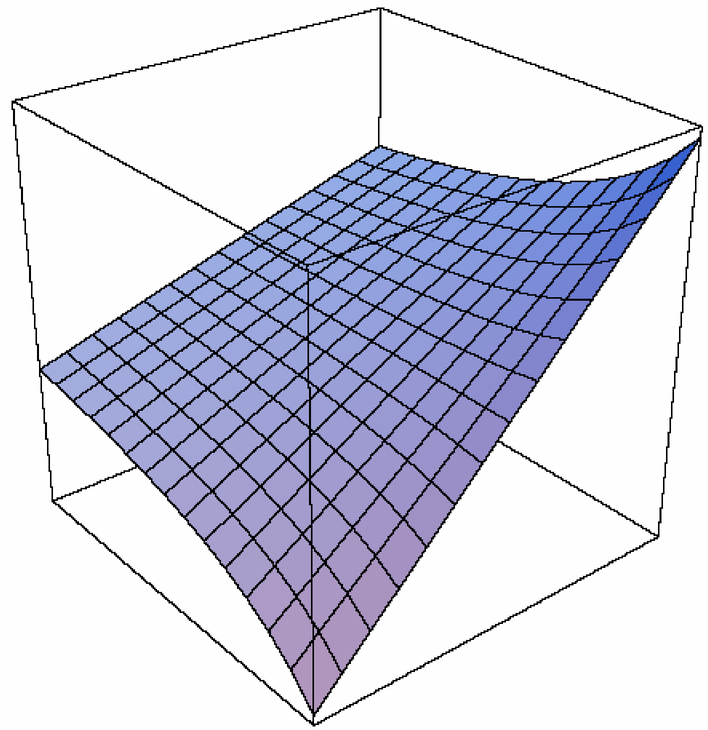




COMMON FUNCTIONS IN MODELING

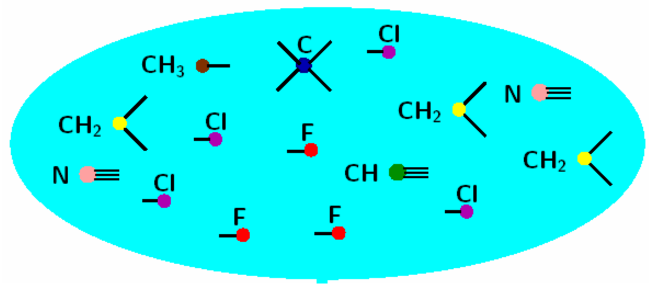


$x1 * x2$

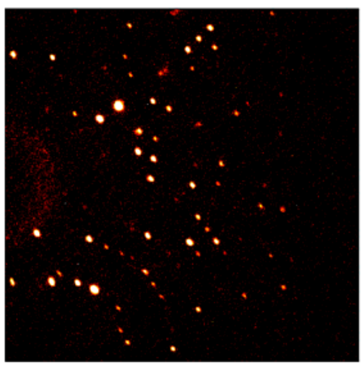


$\frac{x1}{x2}$

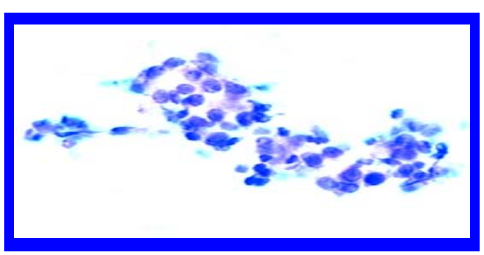
INFORMATICS PROBLEMS IN CHEMISTRY, BIOLOGY, AND MEDICINE



- Design of automotive refrigerants



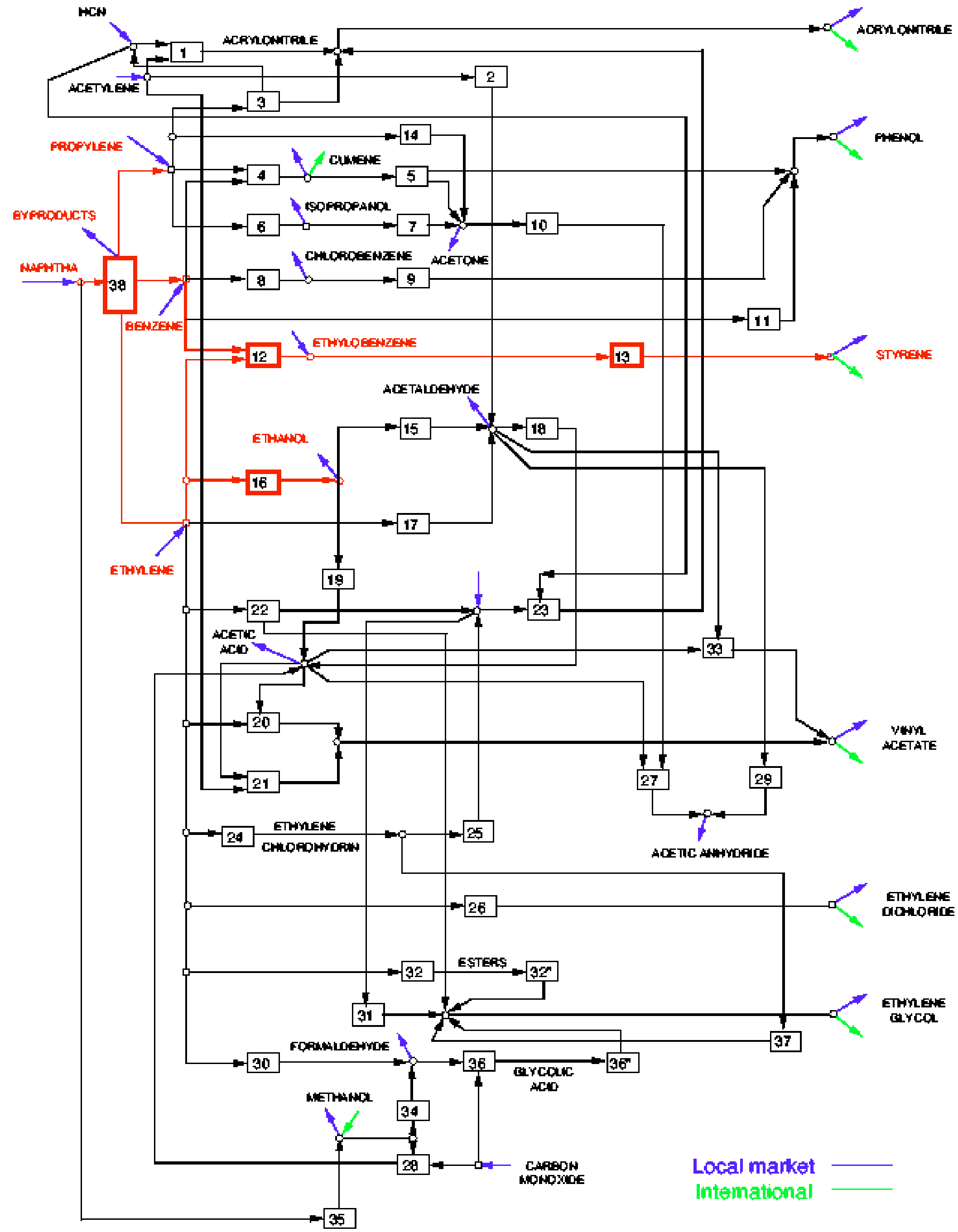
- Molecular structure determination via X-ray crystallography



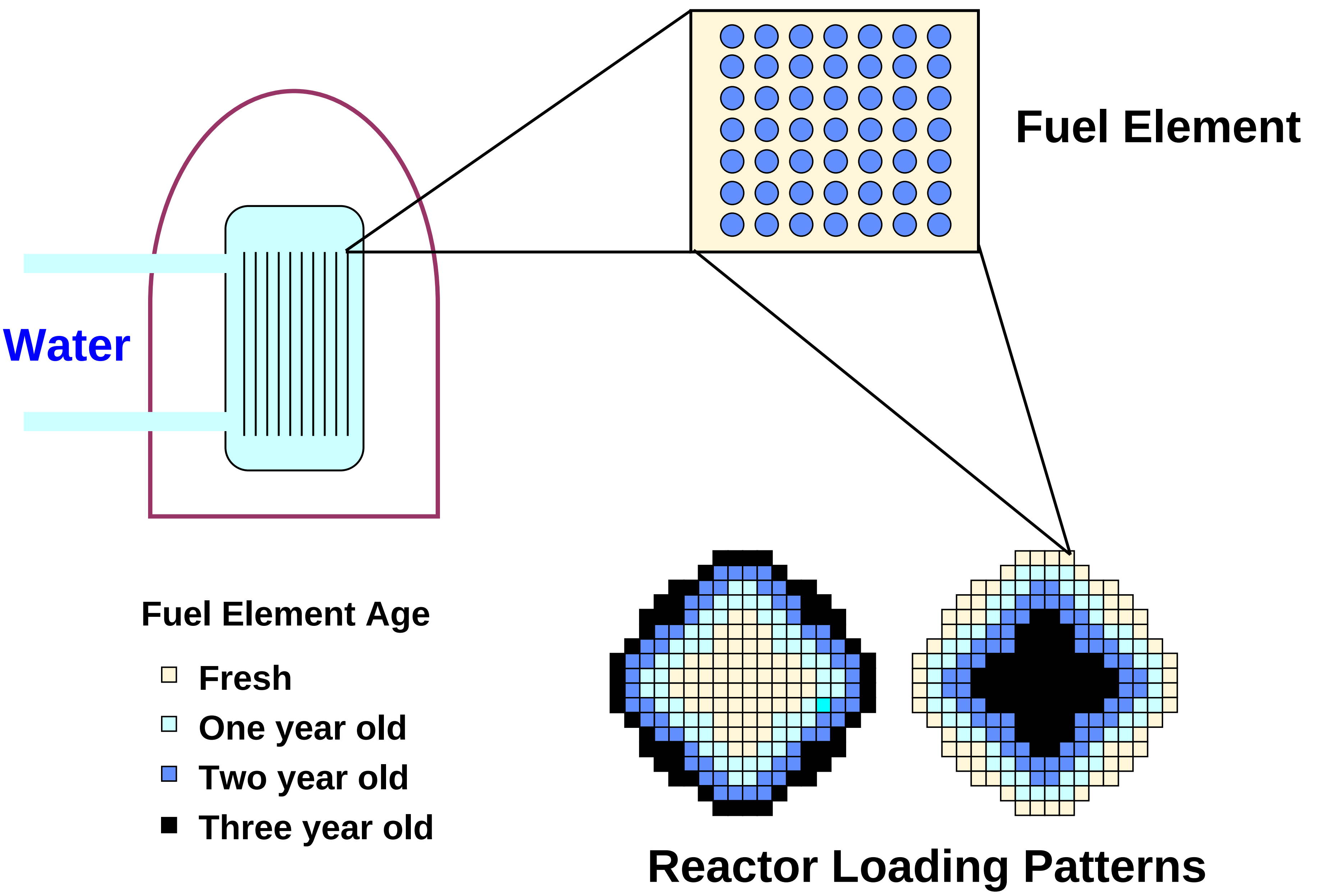
- Breast cancer diagnosis

SUPPLY CHAIN DESIGN & OPERATIONS

- Technology selection
- Facility location
- Capacity expansion
- Blending and pooling
- Uncertainty
- Portfolio optimization
- Very large-scale decision making problems



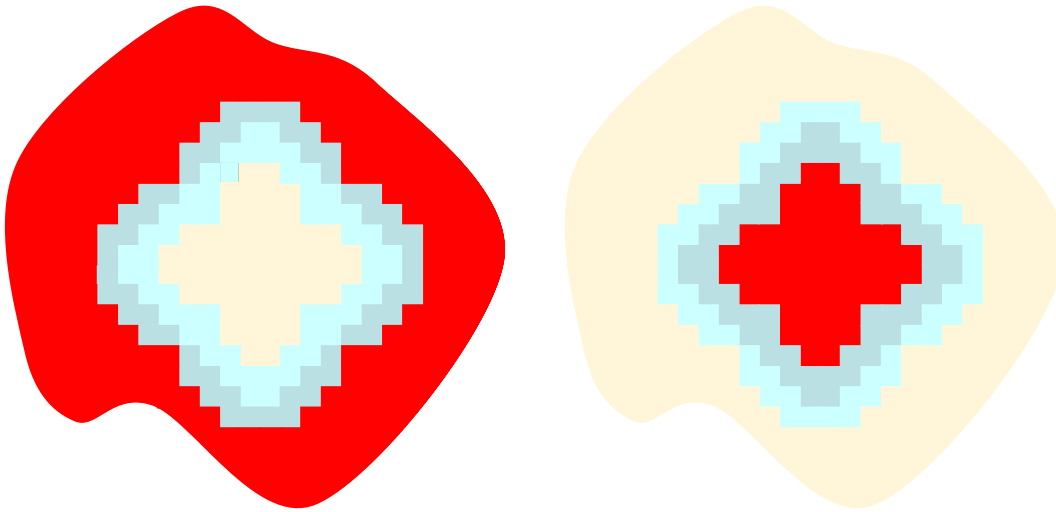
NUCLEAR REACTOR FUEL MANAGEMENT



DRUG RELEASE CONTROL

Drug Concentration

- None
- Low
- Medium
- High



- Which parts of the tumor to cut?
  - Integer decision variables
- What is the optimal concentration profile?
  - Continuous decision variables

APPLICATIONS OF GLOBAL OPTIMIZATION

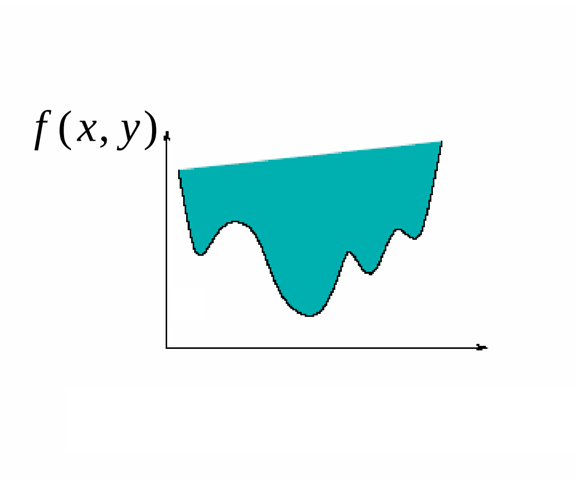
- **Engineering design & manufacturing:**
  - Product & process design
  - Production planning-scheduling-logistics
- **Computational chemical sciences:**
  - Chemical & phase equilibria
  - Molecular design
- **Informatics problems in biology and medicine**
  - Molecular structure prediction
  - Diagnosis

↓

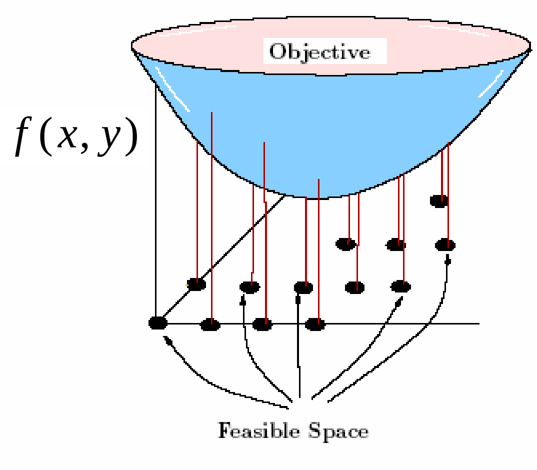
Global optimization is an empowering technology in science and engineering

MIXED-INTEGER NONLINEAR PROGRAMMING

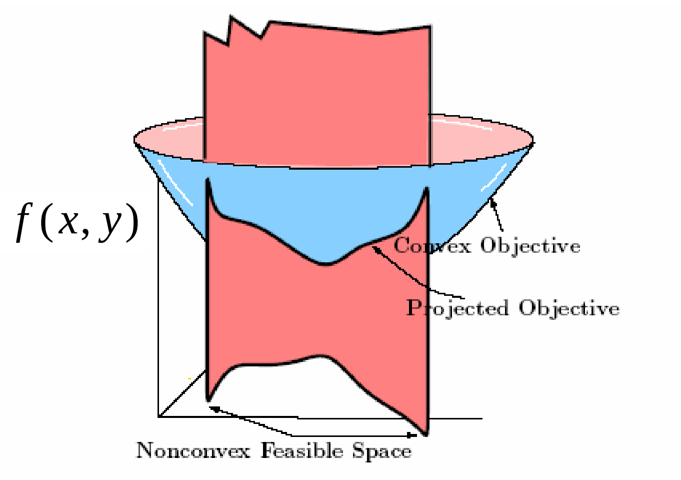
$$\begin{aligned} \min \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0 \\ & x \in R^n, \ \ y \in Z^p \end{aligned}$$



Multimodal objective

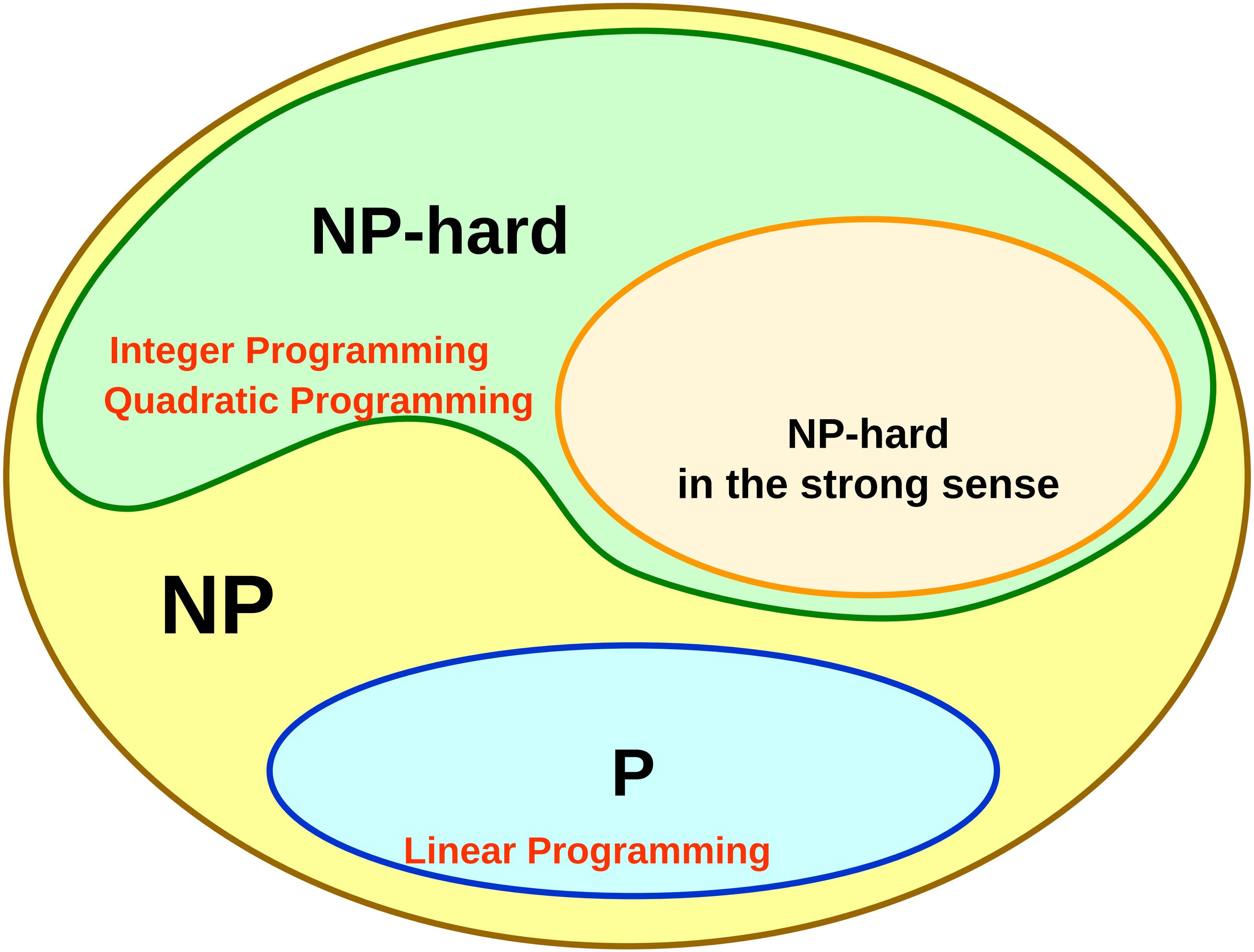


Integrality conditions

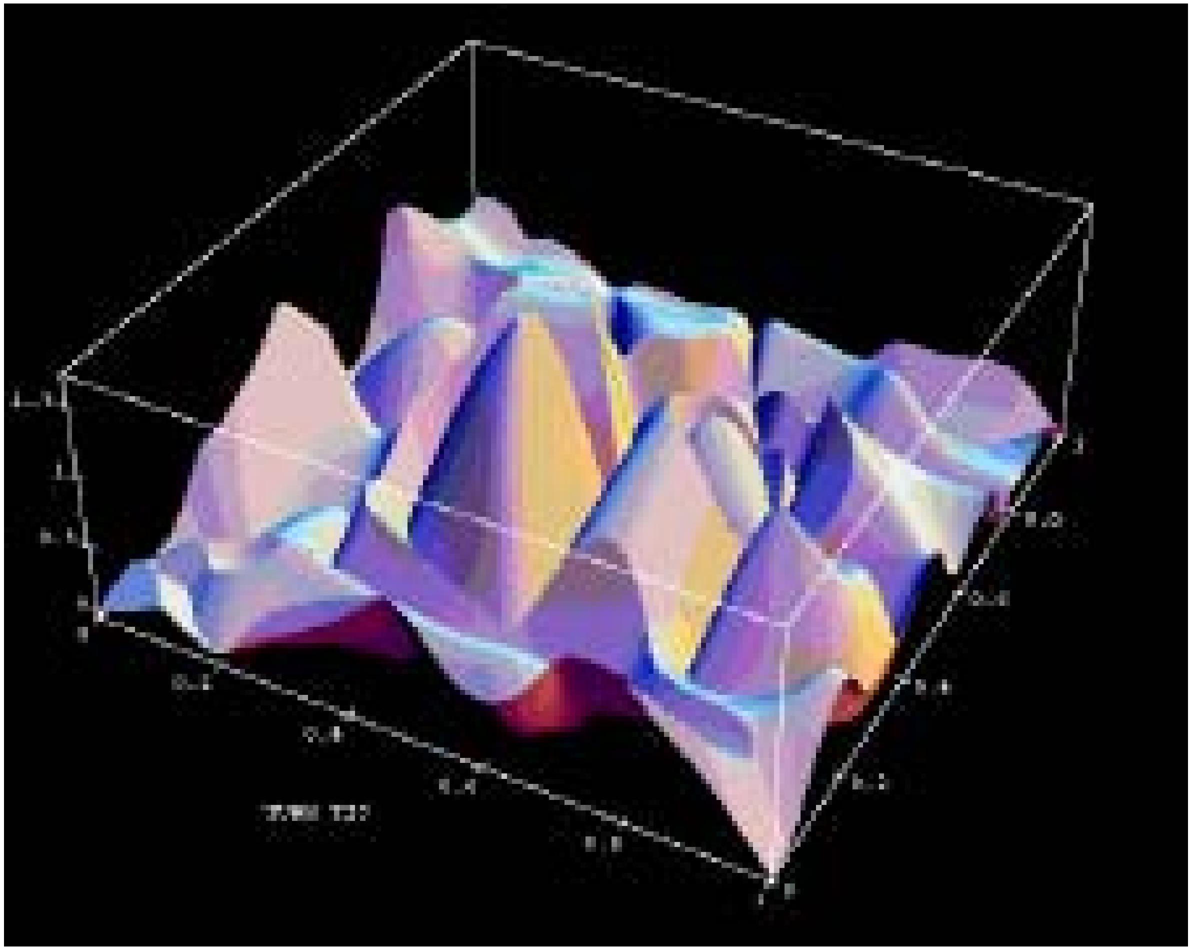


Nonconvex constraints

GLOBAL OPTIMIZATION PROBLEMS ARE NP-HARD

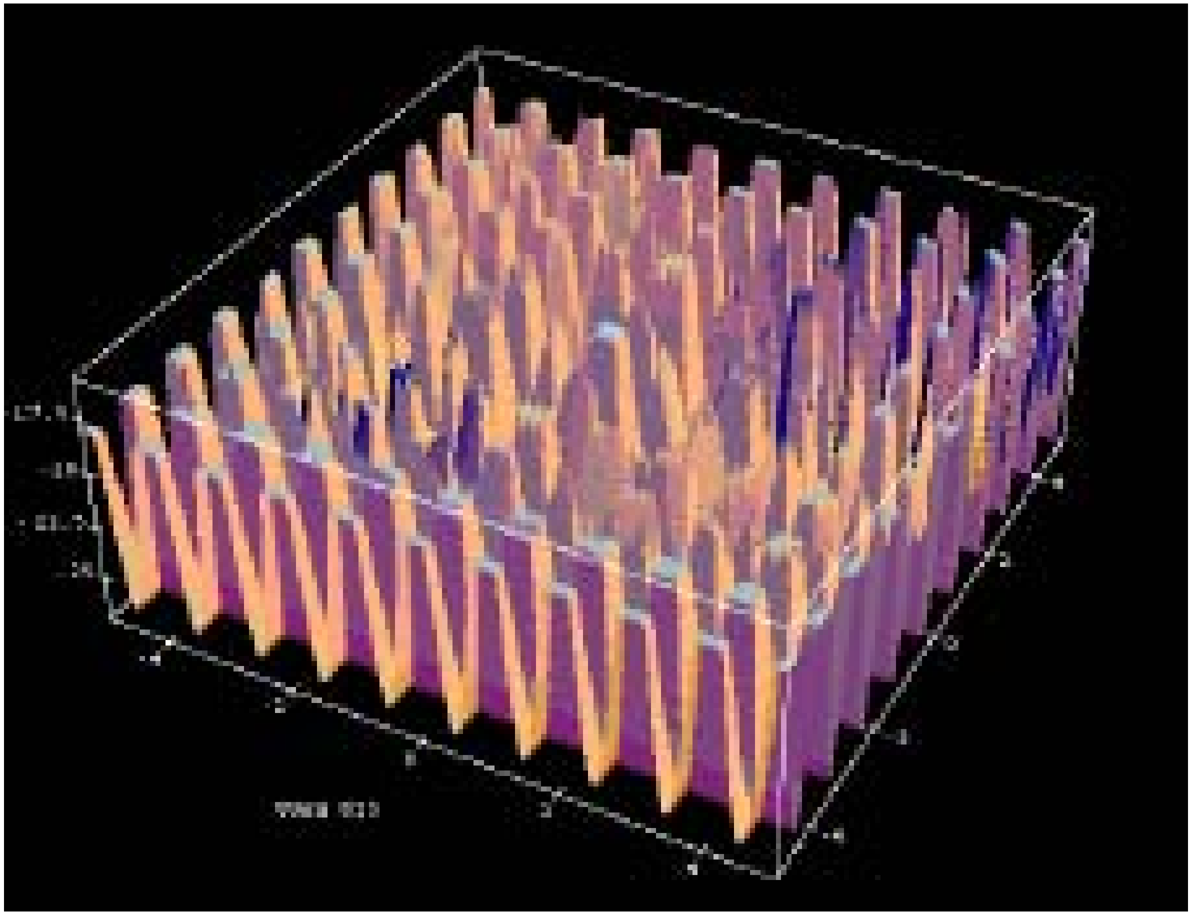


NONCONVEX FUNCTIONS—MOUNTAINS



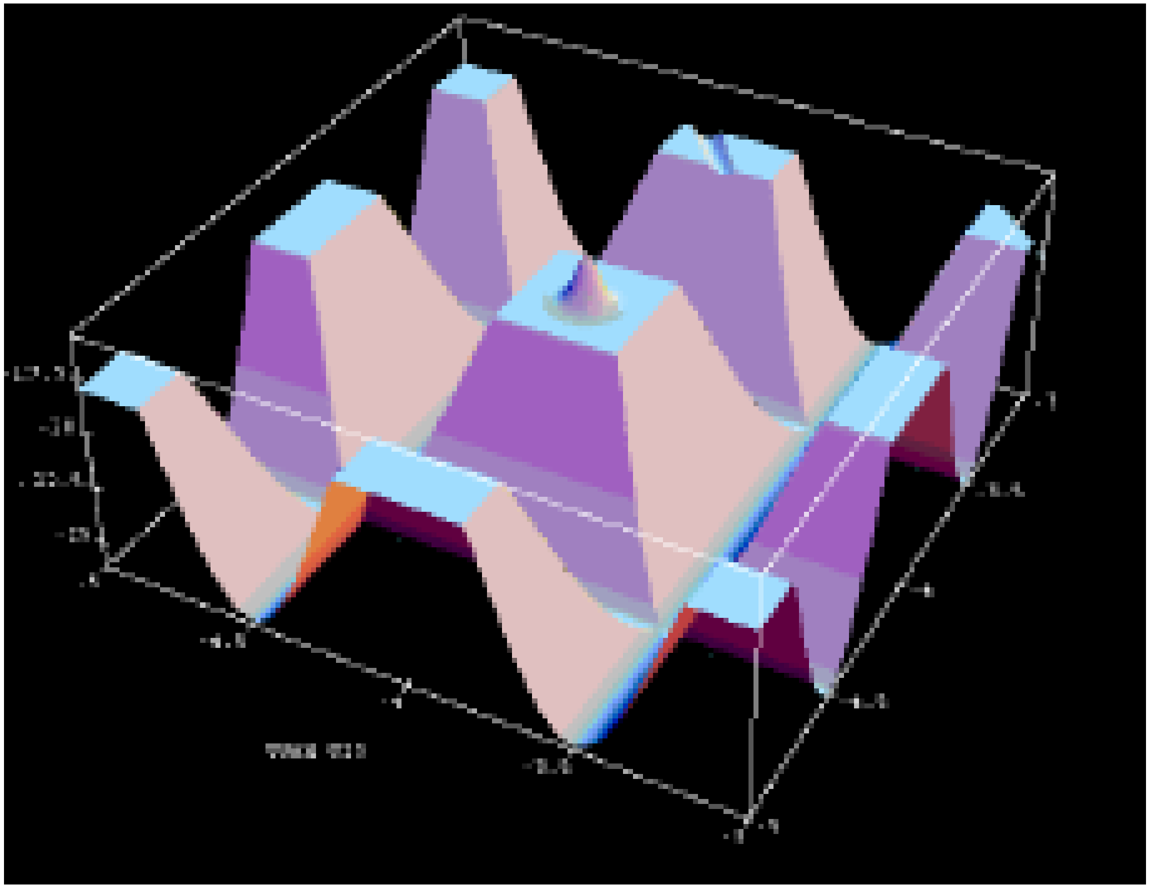
Taken from http://www.ti3.tu-harburg.de/english/globopt.html

NONCONVEX FUNCTIONS—PLATEAUS



Taken from http://www.ti3.tu-harburg.de/english/globopt.html

NONCONVEX FUNCTIONS—DETAILS OF PLATEAUS



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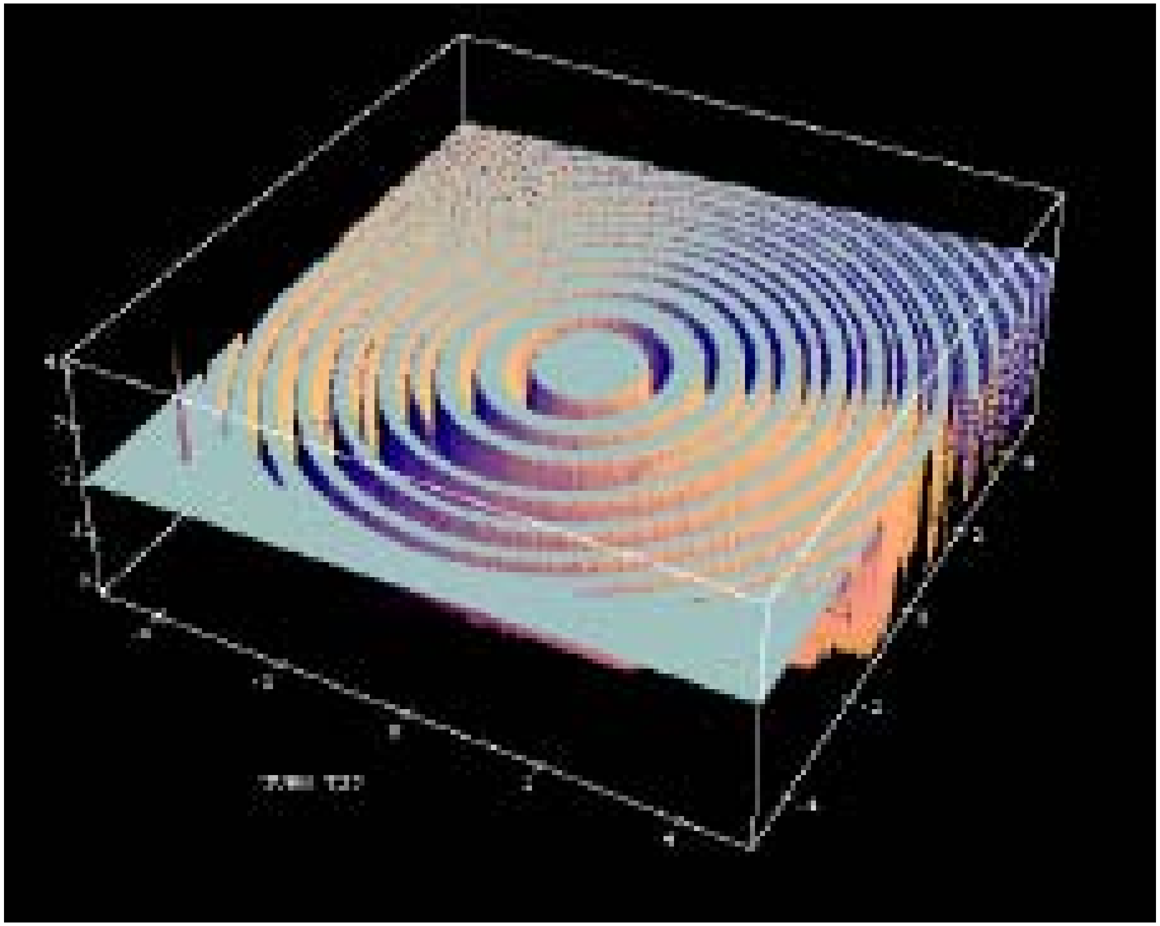
NONCONVEX FUNCTIONS—DARTBAORD WITH ARROW



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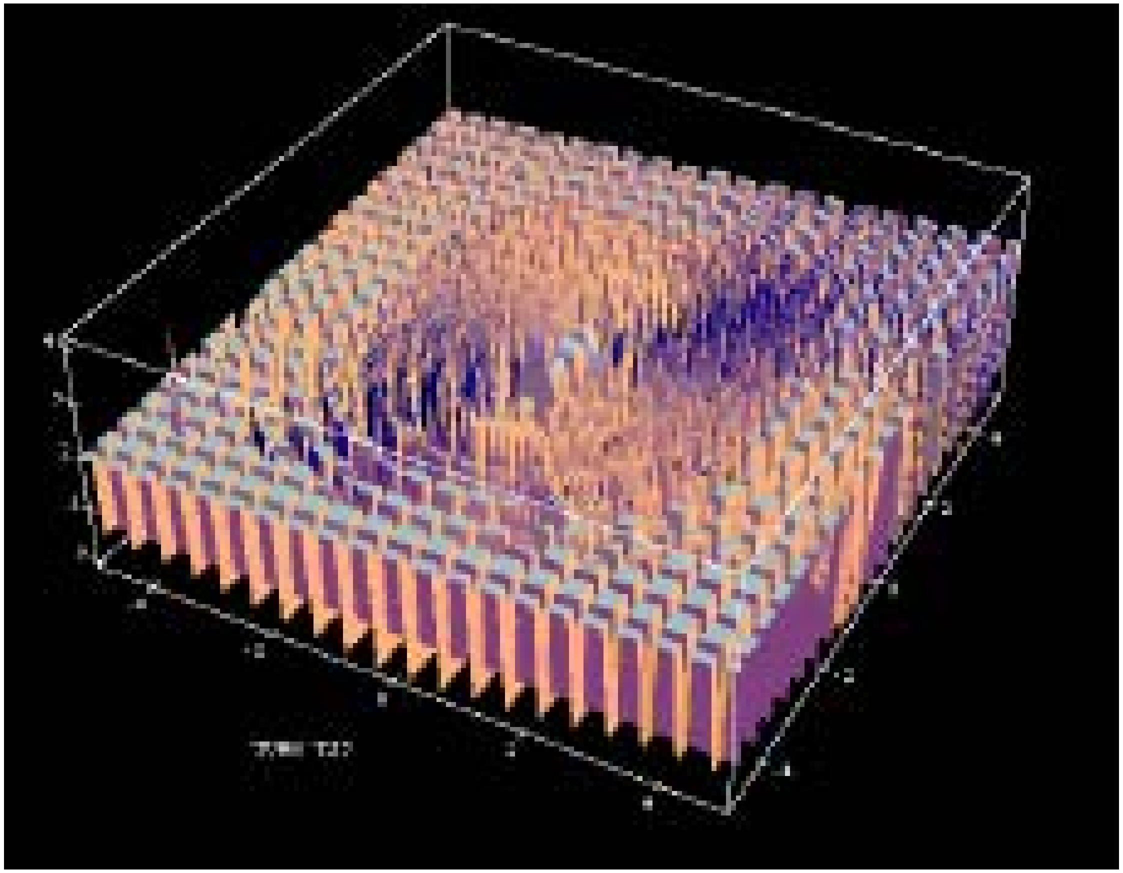
NONCONVEX FUNCTIONS—BRYCE CANYON



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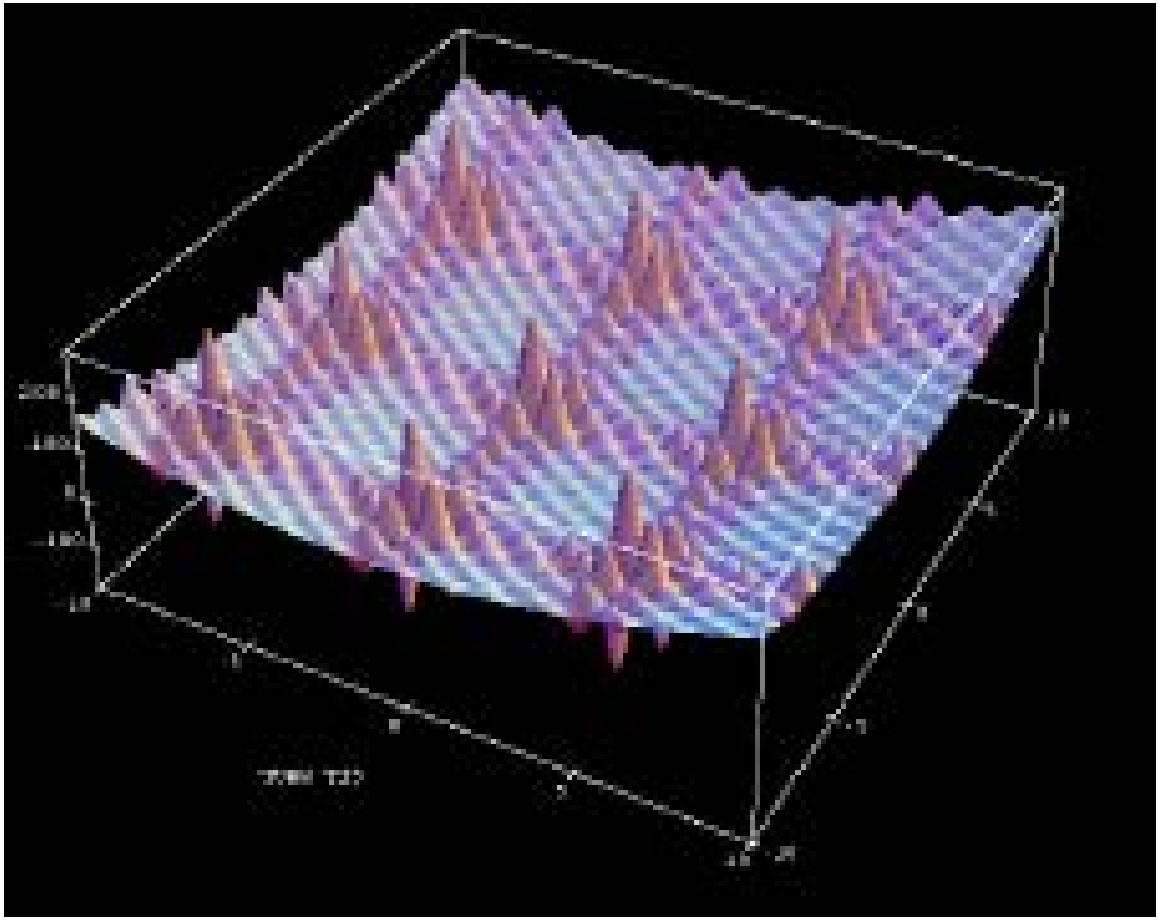
NONCONVEX FUNCTIONS—LEVY'S BENT EGG CARDBOARD



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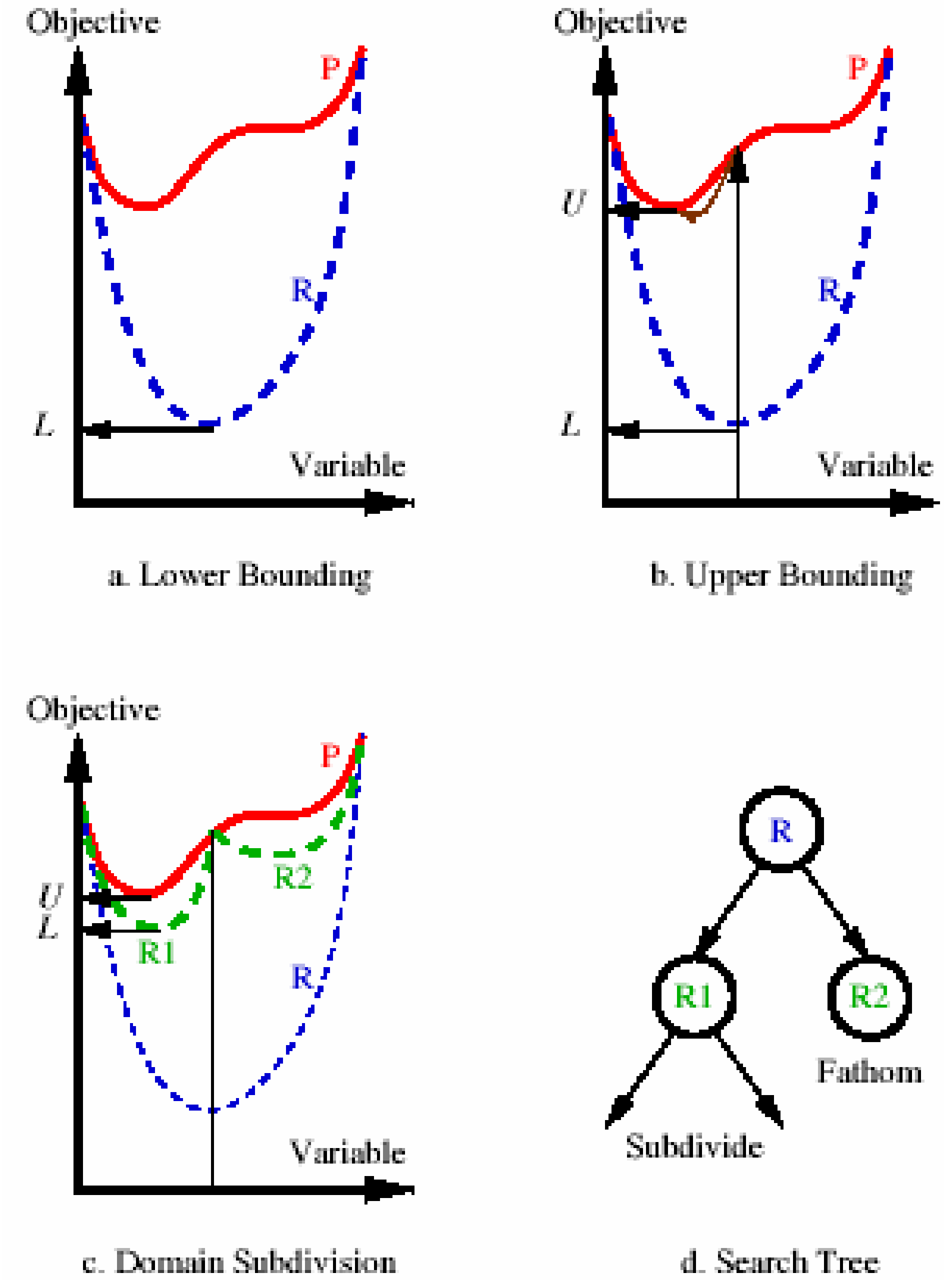
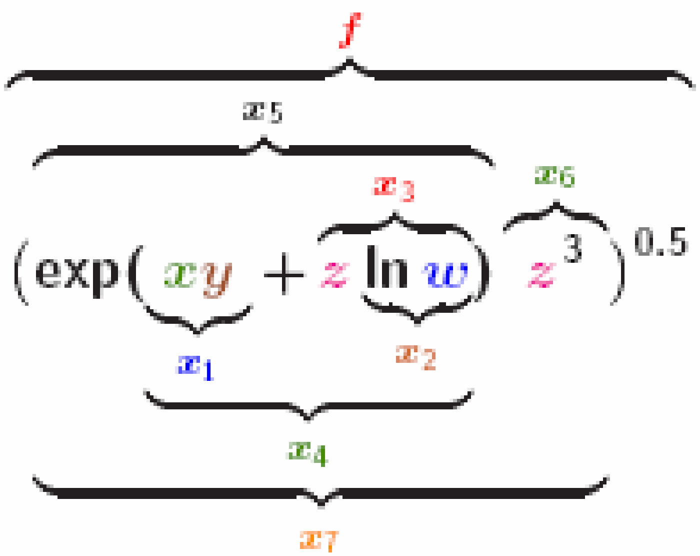
BRANCH-AND-BOUND



PROCEDURE FOR BOUNDING FACTORABLE PROGRAMS

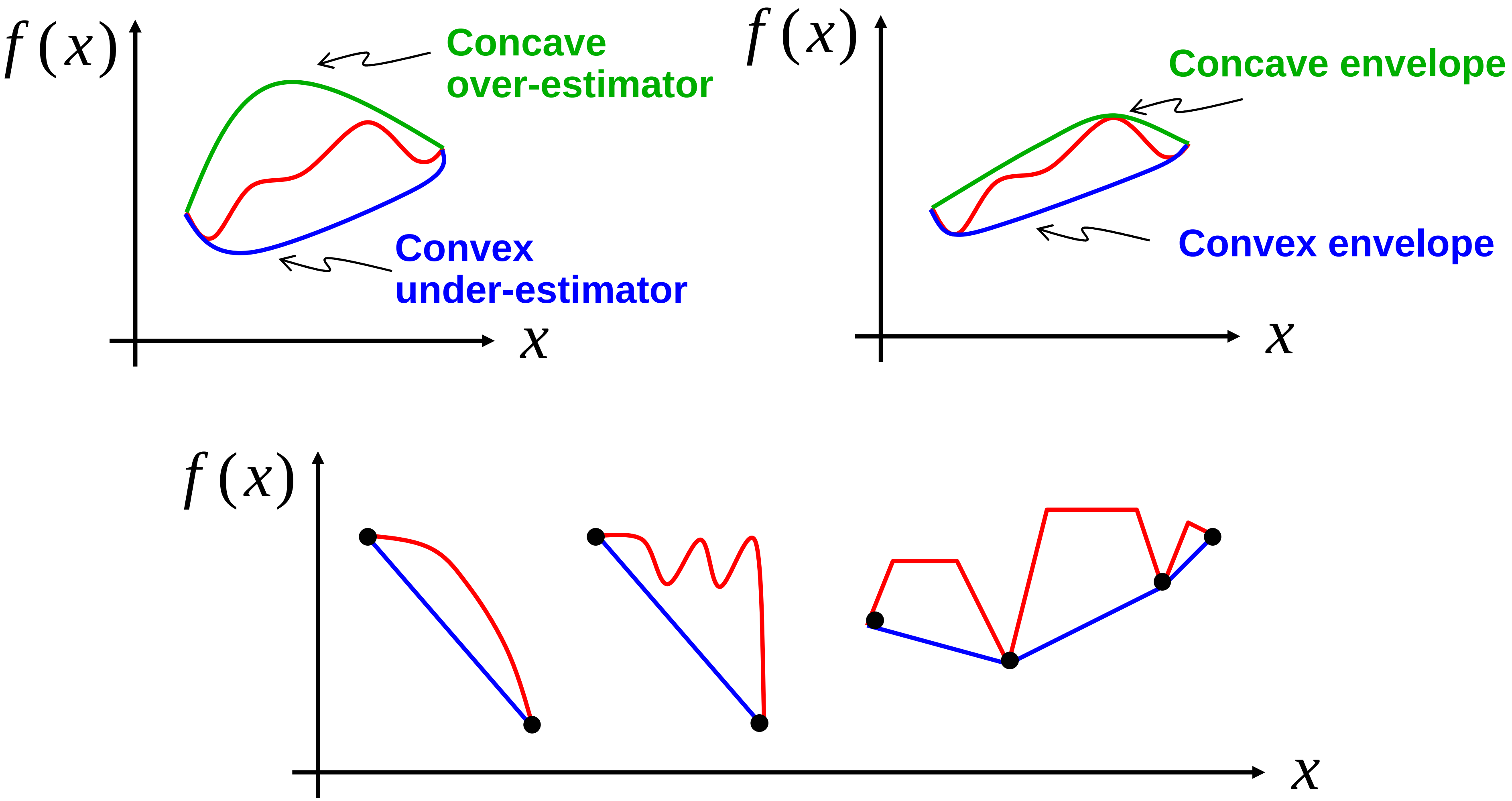
Introduce variables for intermediate quantities whose envelopes are not known

Example: $f(x, y, z, w) = \sqrt{\exp(xy + z \ln w) z^3}$



$$x_1 = xy$$
$$x_2 = \ln(w)$$
$$x_3 = z x_2$$
$$x_4 = x_1 + x_3$$
$$x_5 = \exp(x_4)$$
$$x_6 = z^3$$
$$x_7 = x_5 x_6$$
$$f = \sqrt{x_7}$$

CONVEX/CONCAVE ENVELOPES OFTEN FINITELY GENERATED

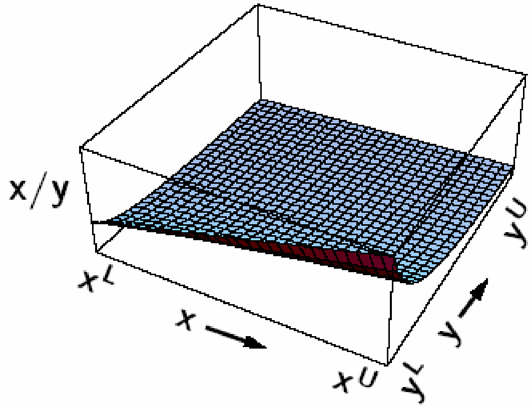


TWO-STEP CONVEX ENVELOPE CONSTRUCTION VIA CONVEX EXENSIONS

1. **Identify generating set** **(Tawarmalani and Sahinidis, 2002):**
   - **Key result:** **A point in set X is *not* in the generating set if it is not in the generating set over a neighborhood of X that contains it**

2. **Use disjunctive programming techniques to construct epigraph over the generating set**
   - **Rockafellar (1970)**
   - **Balas (1974)**

RATIO: FACTORABLE RELAXATION



$$z \geq x/y$$
$$y^L \leq y \leq y^U$$
$$x^L \leq x \leq x^U$$

$z \geq x/y$
$x^L \leq x \leq x^U$
$y^L \leq y \leq y^U$

cross-multiplying →

$zy \geq x$
$y^L \leq y \leq y^U$
$x^L/y^U \leq z \leq x^U/y^L$
$x^L \leq x \leq x^U$

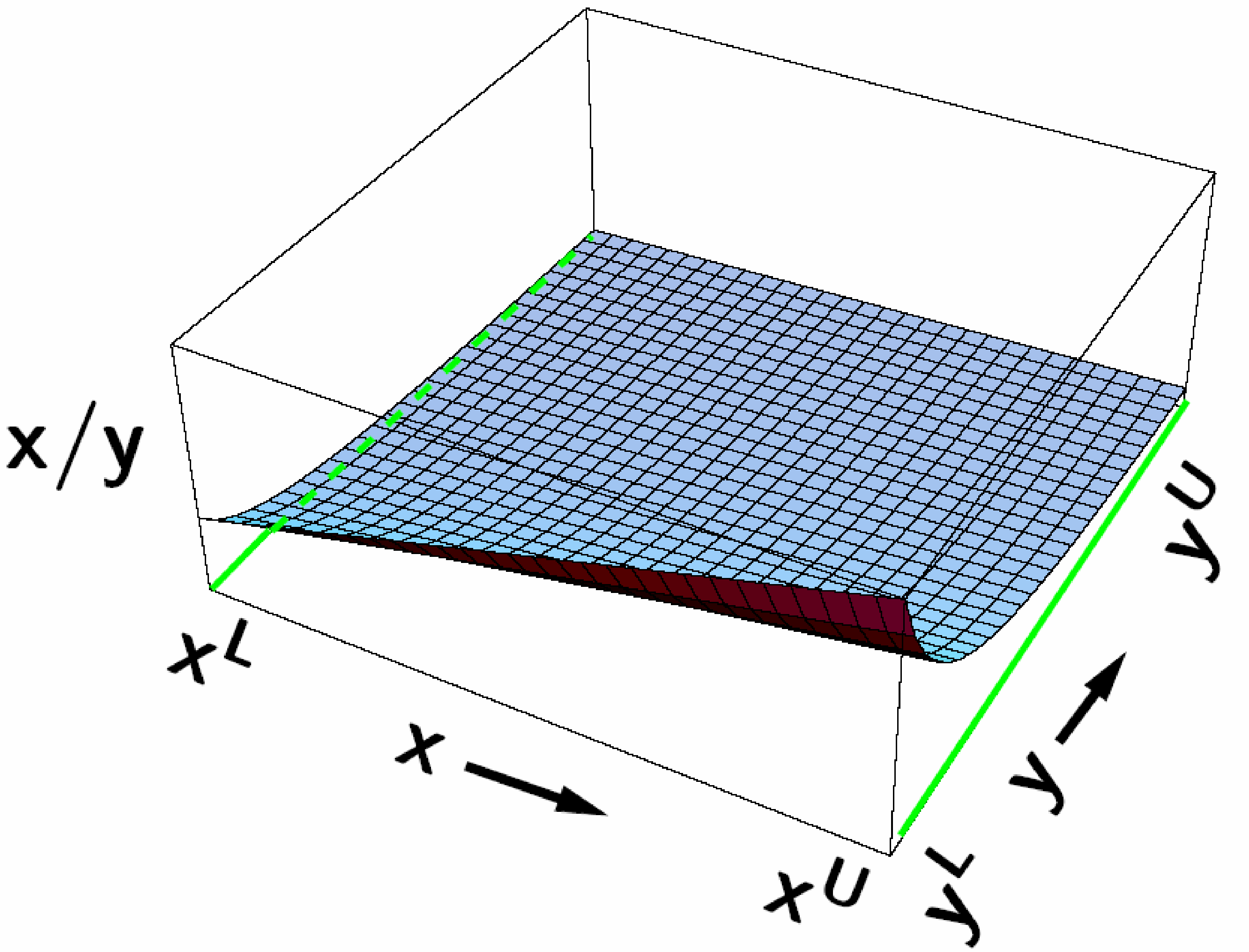
↓ Relaxing

$zy - (z - x^L/y^U)(y - y^U) \geq x$
$zy - (z - x^U/y^L)(y - y^L) \geq x$
$y^L \leq y \leq y^U$
$x^L \leq x \leq x^U$

← Simplifying

$z \geq (xy^U - yx^L + x^L y^U)/{y^U}^2$
$z \geq (xy^L - yx^U + x^U y^L)/{y^L}^2$
$y^L \leq y \leq y^U$
$x^L \leq x \leq x^U$

RATIO: THE GENERATING SET



$$G^{\mathrm{epi}}(x/y) = \left\{ (x, y) \mid x \in \{x^L, x^U\} \right\}$$

ENVELOPES OF MULTILINEAR FUNCTIONS

- **Multilinear function over a box**

$$M(x_1,\ldots,x_n)=\sum_t a_t \prod_{i=1}^{p_t} x_i,\ -\infty < L_i \le x_i \le U_i < +\infty,\ i=1,\ldots,n$$

- **Generating set**

$$\mathrm{vert}\left(\prod_{i=1}^{n}[L_i,U_i]\right)$$

- **Polyhedral convex encloser follows trivially from polyhedral representation theorems**

PRODUCT DISAGGREGATION

Consider the function:

$$\phi(x; y_1, \ldots, y_n) = a_0 + \sum_{k=1}^{n} a_k y_k + x b_0 + x \sum_{k=1}^{n} b_k y_k$$

Let

$$H = [x^L, x^U] \times \prod_{k=1}^{n} [y_k^L, y_k^U]$$

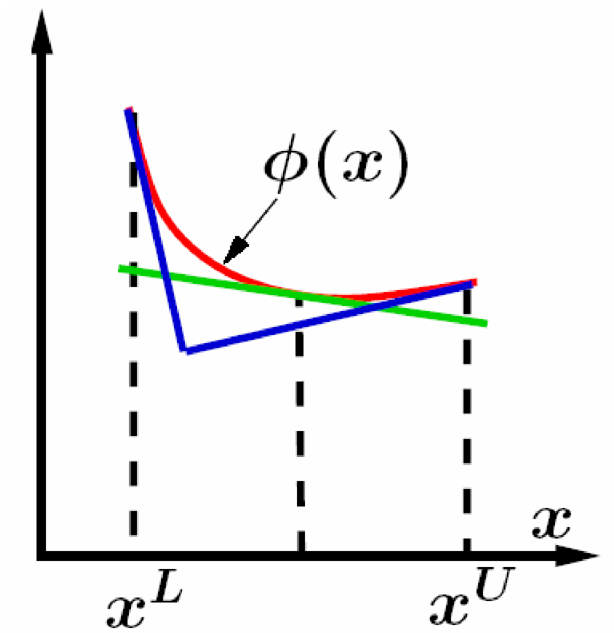
Then

$$\text{convenv}_H \, \phi = \quad a_0 + \sum_{k=1}^{n} a_k y_k + x b_0 + \sum_{k=1}^{n} \text{convenv}_{[y_k^L, y_k^U] \times [x^L, x^U]} (b_k y_k x)$$

Disaggregated formulations are tighter

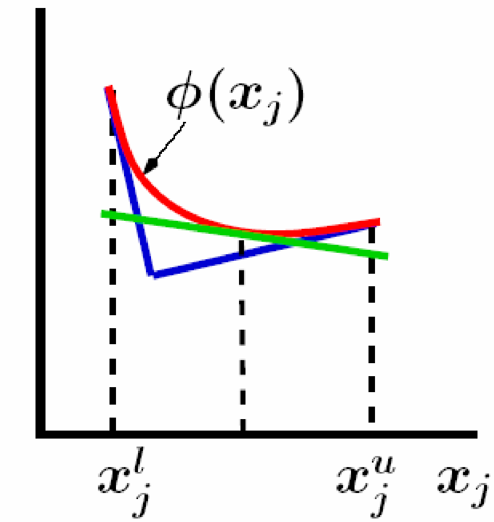
POLYHEDRAL OUTER APPROXIMATION

- Convex NLP solvers are not as robust as LP solvers
- Linear programs can be solved efficiently
- Outer-approximate convex relaxation by polyhedron

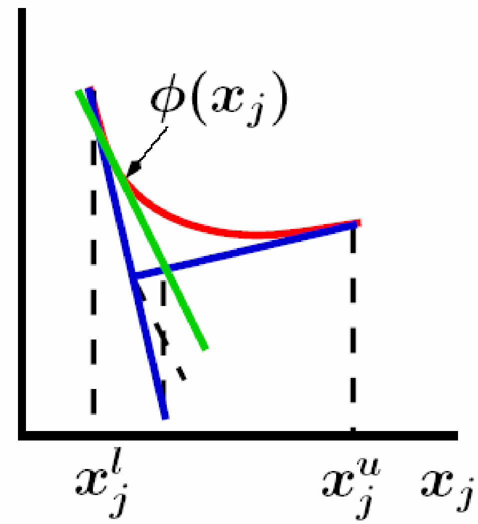


Enjoys quadratic convergence

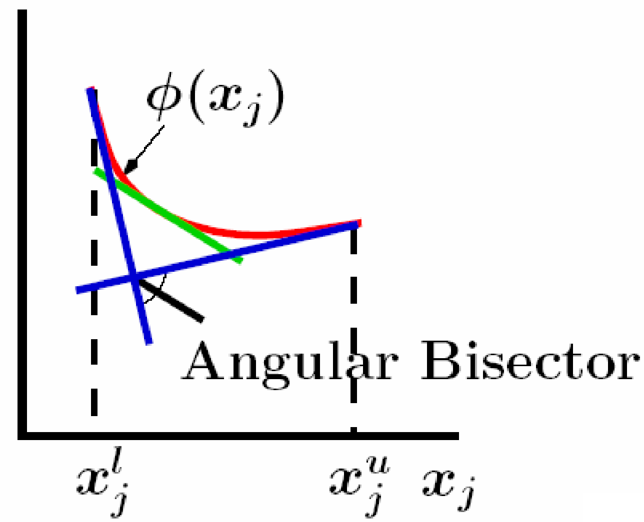
TANGENT LOCATION RULES



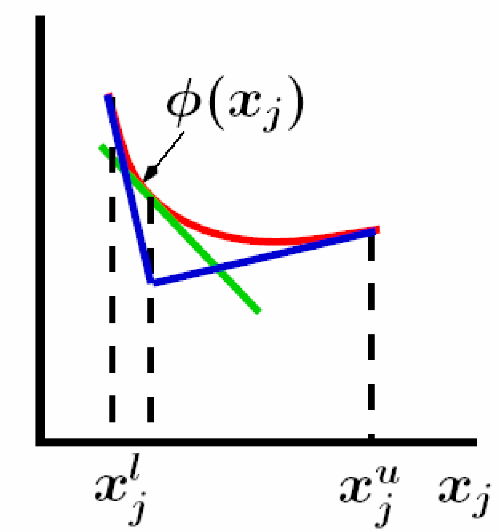
Interval bisection



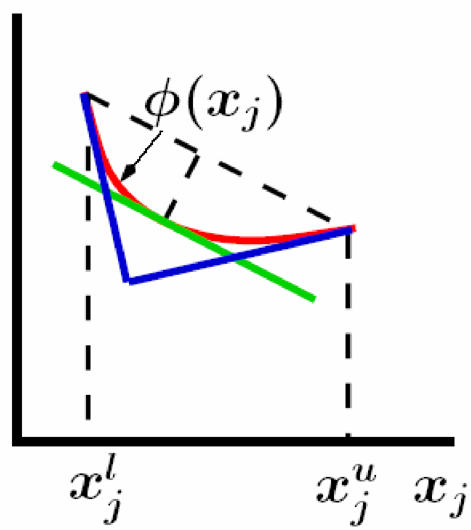
Slope Bisection



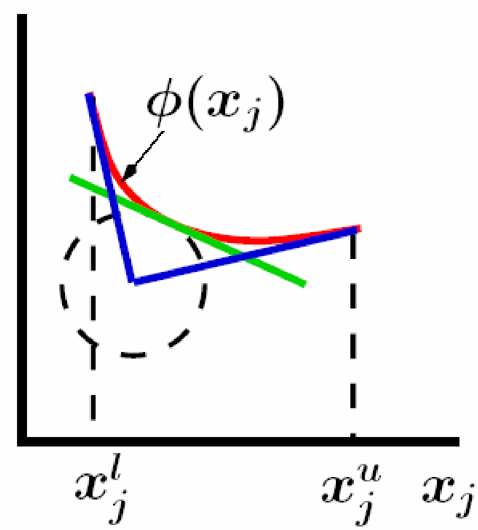
Angle bisection



Maximum error rule

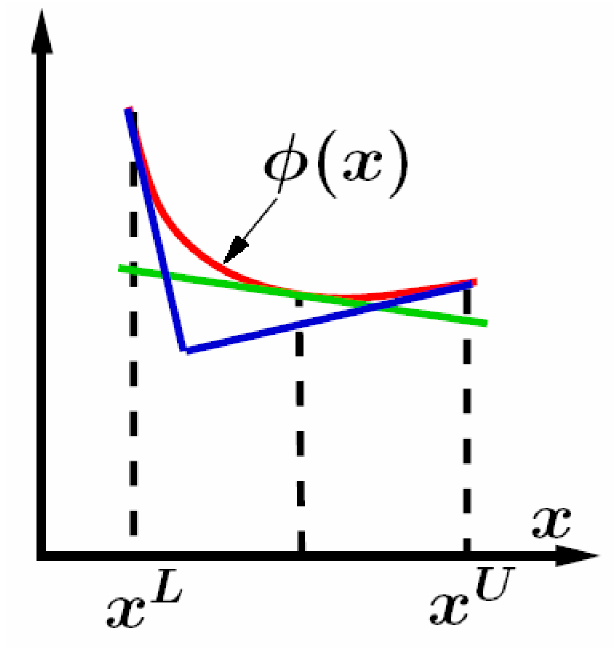


Chord rule



Maximum projective error

EXPLOITING CONVEXITY



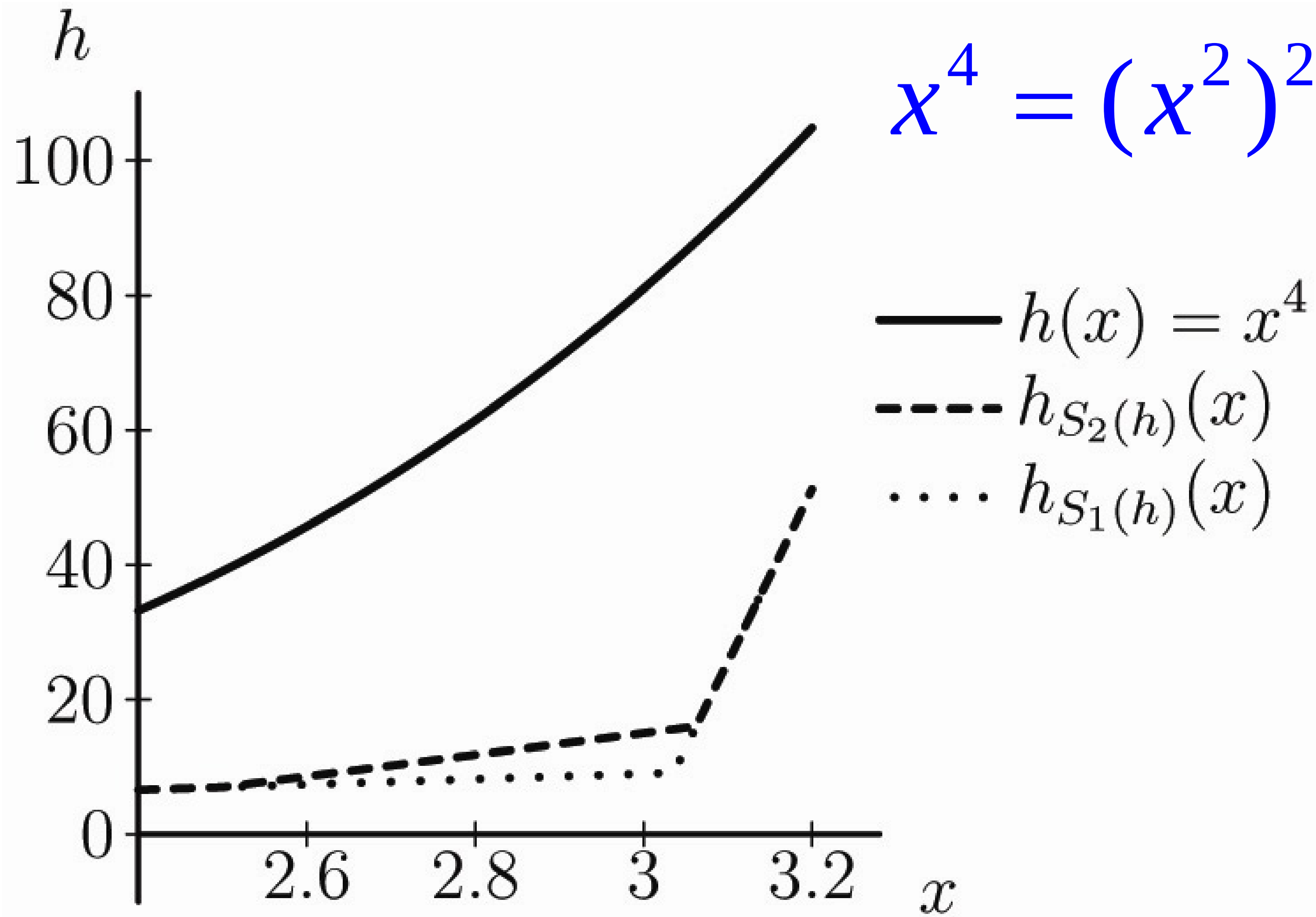
- **Polyhedral relaxations of univariate functions facilitate reliable lower bounding via fast LP routines**
- **Outstanding issues:**
  - **Lower bound itself weakens**
  - **Effect of functional decomposition**
  - **Polyhedral relaxations of convex multivariate functions**
    - **Gruber (1993), Böröczky and Reitzner (2004)**

RECURSIVE FUNCTIONAL COMPOSITIONS

- **Consider *h*=*g*(*f*), where**
  - ***g* and *f* are multivariate convex functions**
  - ***g* is non-decreasing in the range of each nonlinear component of *f***
- ***h* is convex**
- **Two outer approximations of the composite function *h*:**
  - **S1: a single-step procedure that constructs supporting hyperplanes of *h* at a predetermined number of points**
  - **S2: a two-step procedure that constructs supporting hyperplanes for *g* and *f* at corresponding points**
- **Theorem: S2 is contained in S1**
  - **If *f* is affine, S2=S1**
  - **In general, the inclusion is strict**

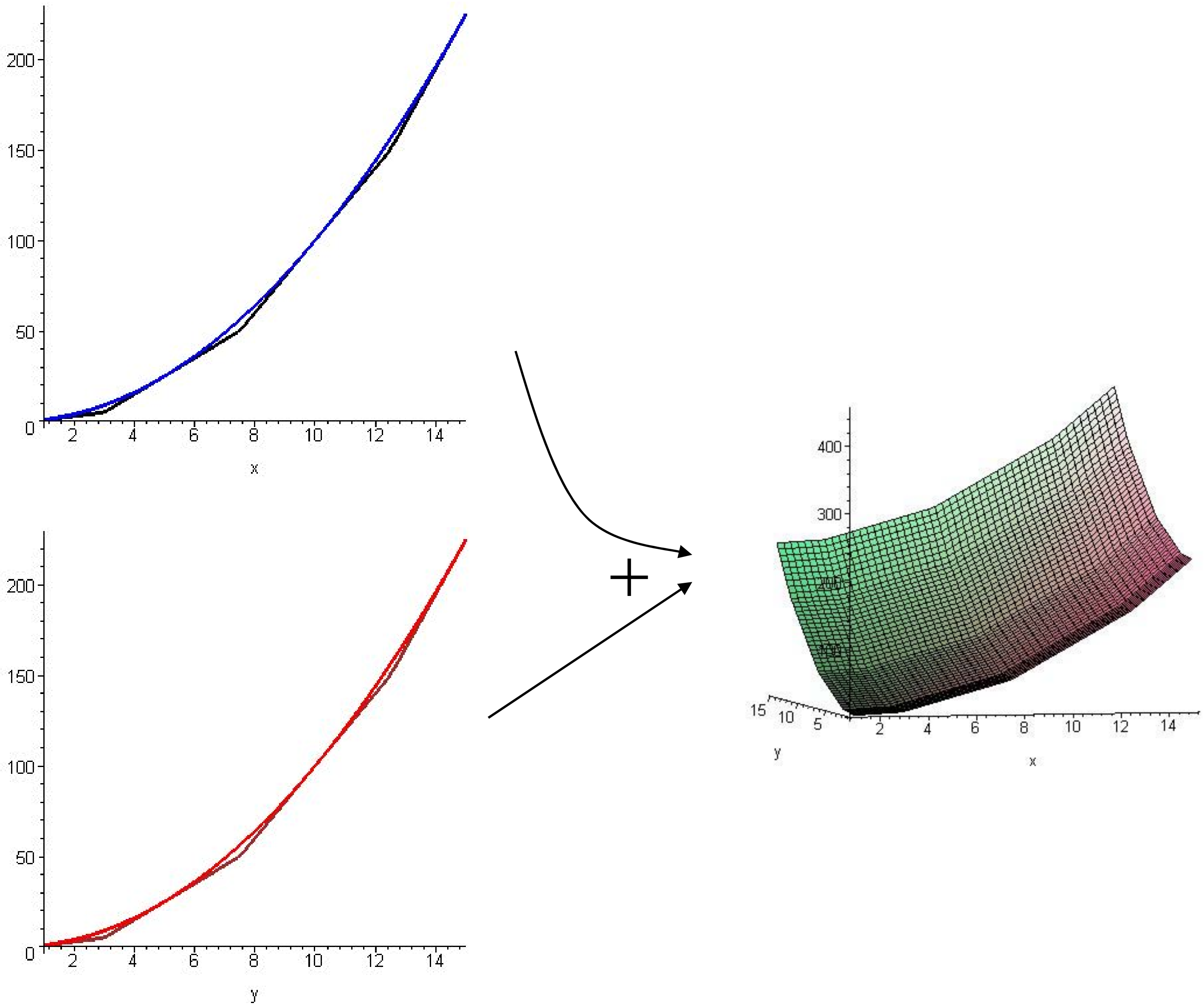
STRICT INCLUSION

$x^4 = (x^2)^2$



Outer - approximating x^4 at $x = 1$ and $x = 4$

OUTER APPROXIMATION OF x^2+y^2



TWO-STEP IS BETTER

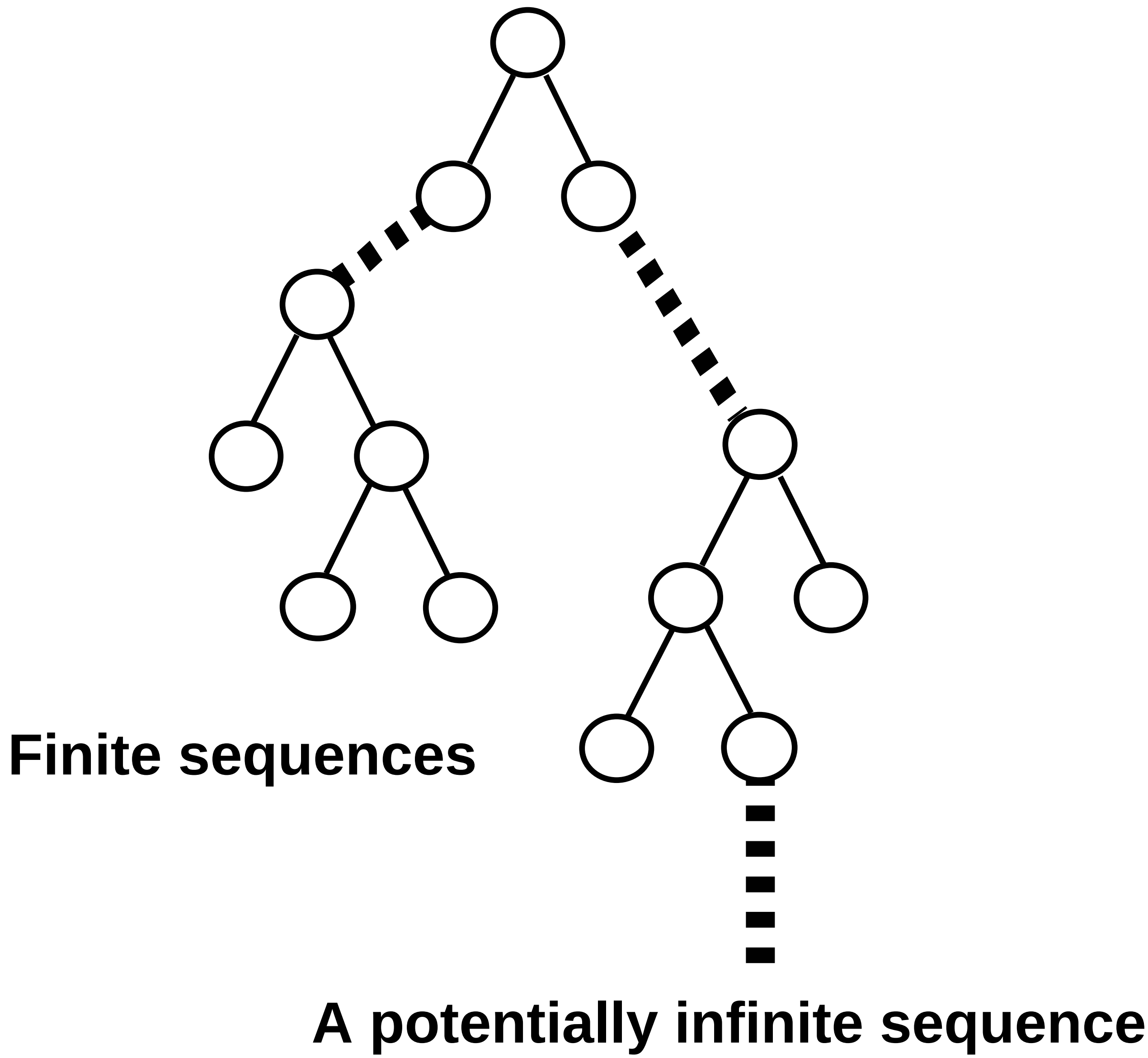
- **Theorem: An exponential number of supporting hyperplanes in S1 may be required to cover S2**

  $h = f_1(x_1) + \ldots + f_m(x_m)$ where each f_i is strictly convex
- **Separable functions are quite common in nonconvex optimization**
- **S2 has the potential of providing much tighter polyhedral outer approximations than S1 with a comparable number of supporting hyperplanes**

AUTOMATIC DETECTION AND EXPLOITATION OF CONVEXITY

- **Composition rule: $h = g(f)$, where**
  - ***g* and *f* are multivariate convex functions**
  - ***g* is non-decreasing in the range of each nonlinear component of *f***
- **Subsumes many known rules for detecting convexity/concavity**
  - ***g* univariate convex, *f* linear**
  - **$g=\max\{f_1(x), \ldots, f_m(x)\}$, each f_i convex**
  - **$g=\exp(f(x))$**
  - **…**
- **Automatic exploitation of convexity is not essential for constructing polyhedral outer approximations in these cases**
  - **However, $\text{logexp}(x) = \log(e^{x_1} + \ldots + e^{x_n})$**
  - **CONVEX_EQUATIONS modeling language construct**

FINITE VERSUS CONVERGENT BRANCH-AND-BOUND ALGORITHMS

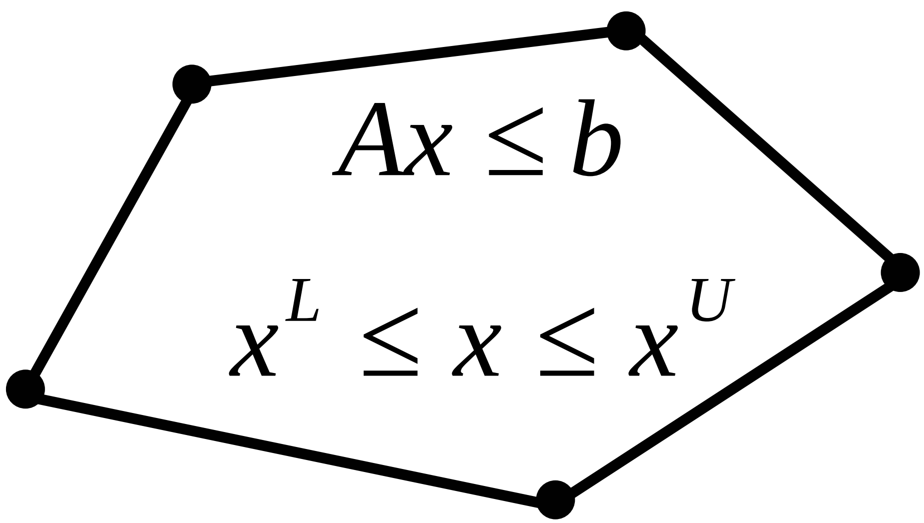


SEPARABLE CONCAVE MINIMIZATION

Consider $f_k(x_k)$ concave, $\forall k.$

$$\min\ f(x) = \sum_k f_k(x_k)$$

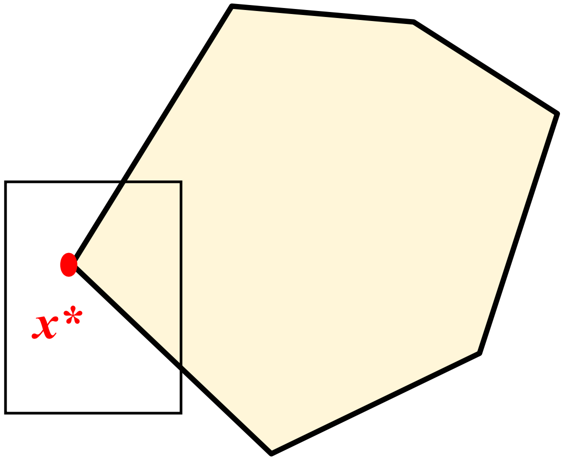
s.t.

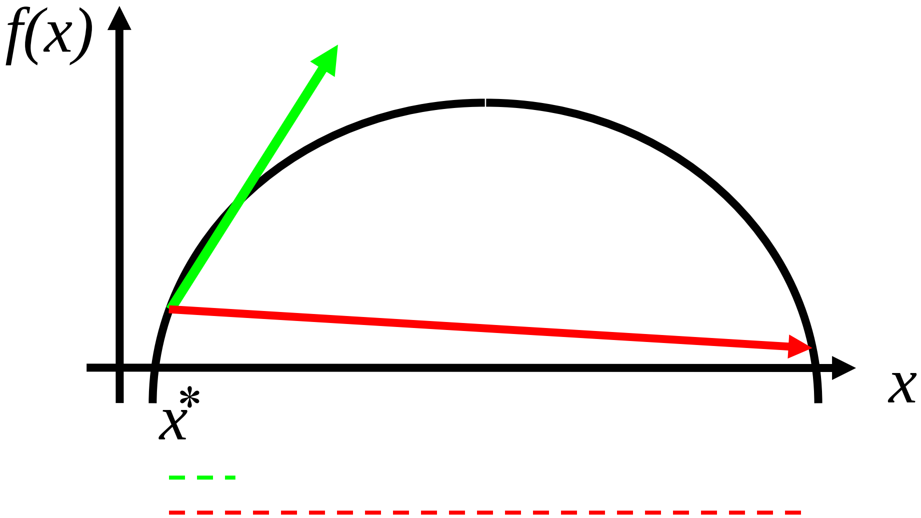


$$Ax \le b$$

$$x^L \le x \le x^U$$

FINITE BRANCHING RULE





- **Branching variable selection:**
  - **Typically, select variable with largest underestimating gap**
  - **Occasionally, select variable corresponding to largest edge**
- **Branching point selection:**
  - **Typically, at the midpoint (exhaustiveness)**
  - **When possible, at the best currently known solution**
    - **Makes underestimators exact at the candidate solutions**
- **Finite isolation of global optimum**
- **Ascend directions of LP also ascend directions of concave program**

STOCHASTIC INTEGER PROGRAMMING

$$\min \quad f(x_1, x_2) = -1.5x_1 - 4x_2 + \sum_{s=1}^{4} \frac{1}{4} Q^s(x_1, x_2)$$

$$s.t. \quad 0 \le x_1, x_2 \le 5$$

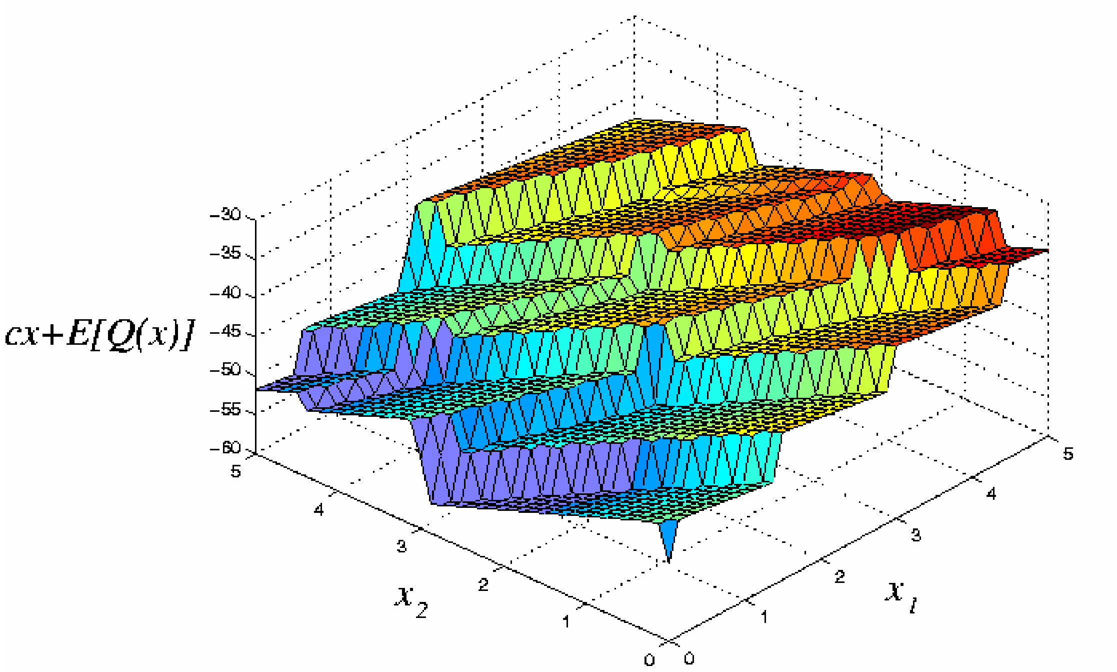
$$Q^s(x_1, x_2) := \quad \min \quad -16y_1 - 19y_2 - 23y_3 - 28y_4$$

$$s.t. \quad 2y_1 + 3y_2 + 4y_3 + 5y_4 \le \omega_1^s - \frac{1}{2}x_1 - \frac{2}{3}x_2$$

$$6y_1 + y_2 + 3y_3 + 2y_4 \le \omega_2^s - \frac{2}{3}x_1 - \frac{1}{3}x_2$$

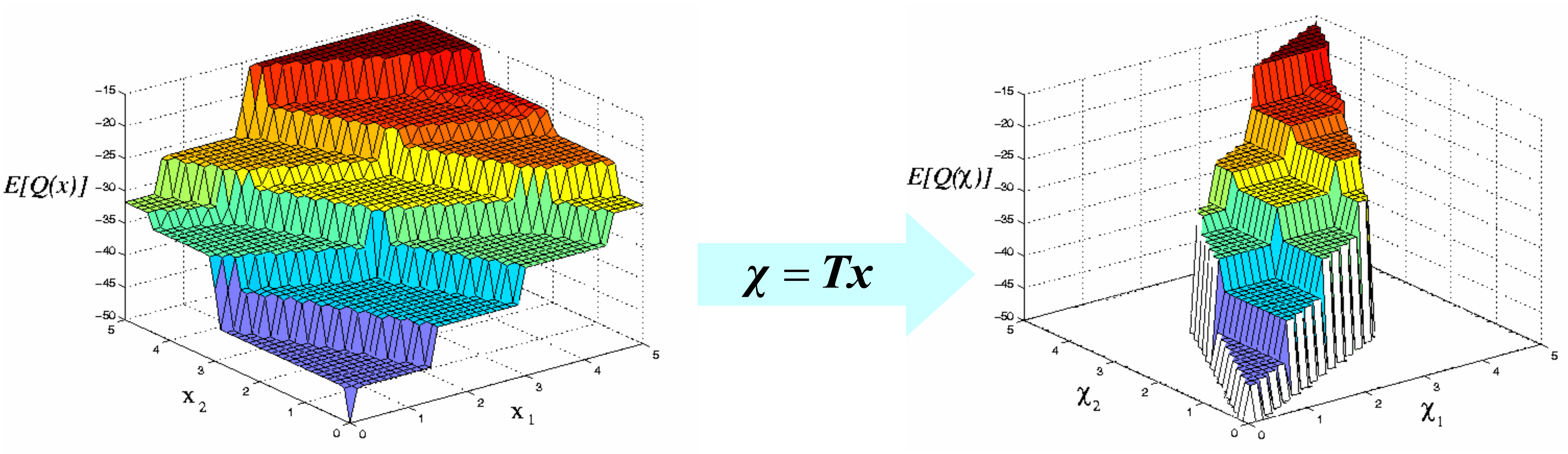
$$y_i \in \{0,1\} \quad \text{for } i = 1,\ldots,4$$

where $(\omega_1, \omega_2) \in \{5,15\} \times \{5,15\}$



- **Discontinuous**
- **Highly non-convex**
- **Many local minima**

VARIABLE TRANSFORMATION

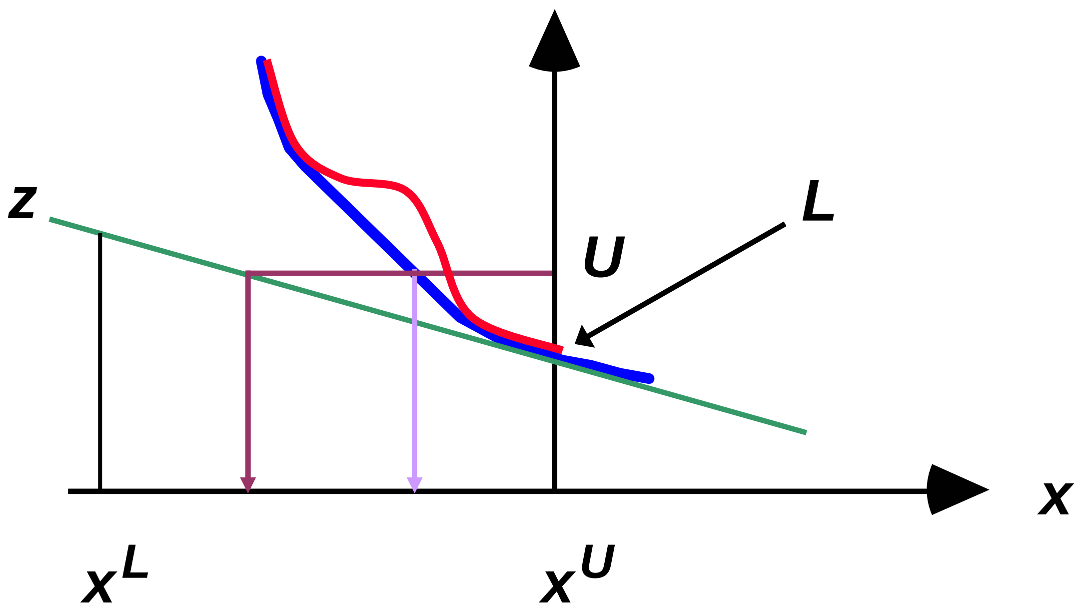


Ahmed, Tawarmalani and Sahinidis (2004)

- **Solve the problem in the space of the "tender variables"**
- **Variable transformation aligns discontinuities orthogonal to variable axes**
- **Discontinuities identified based on Blair and Jeroslow (1977) results**
- **Finite termination**

MARGINALS-BASED RANGE REDUCTION

Relaxed Value Function



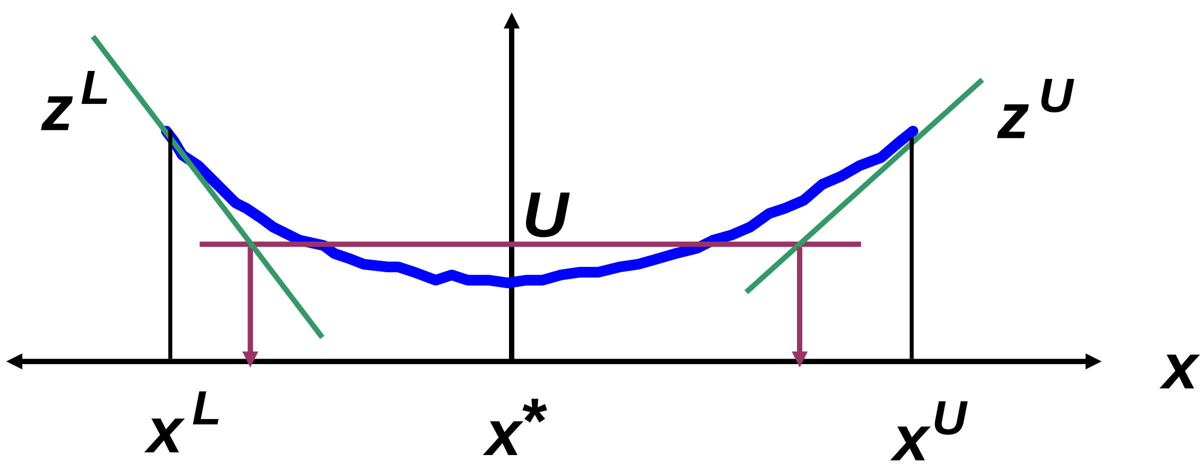
If a variable goes to its upper bound at the relaxed problem solution, this variable's lower bound can be improved

PROBING

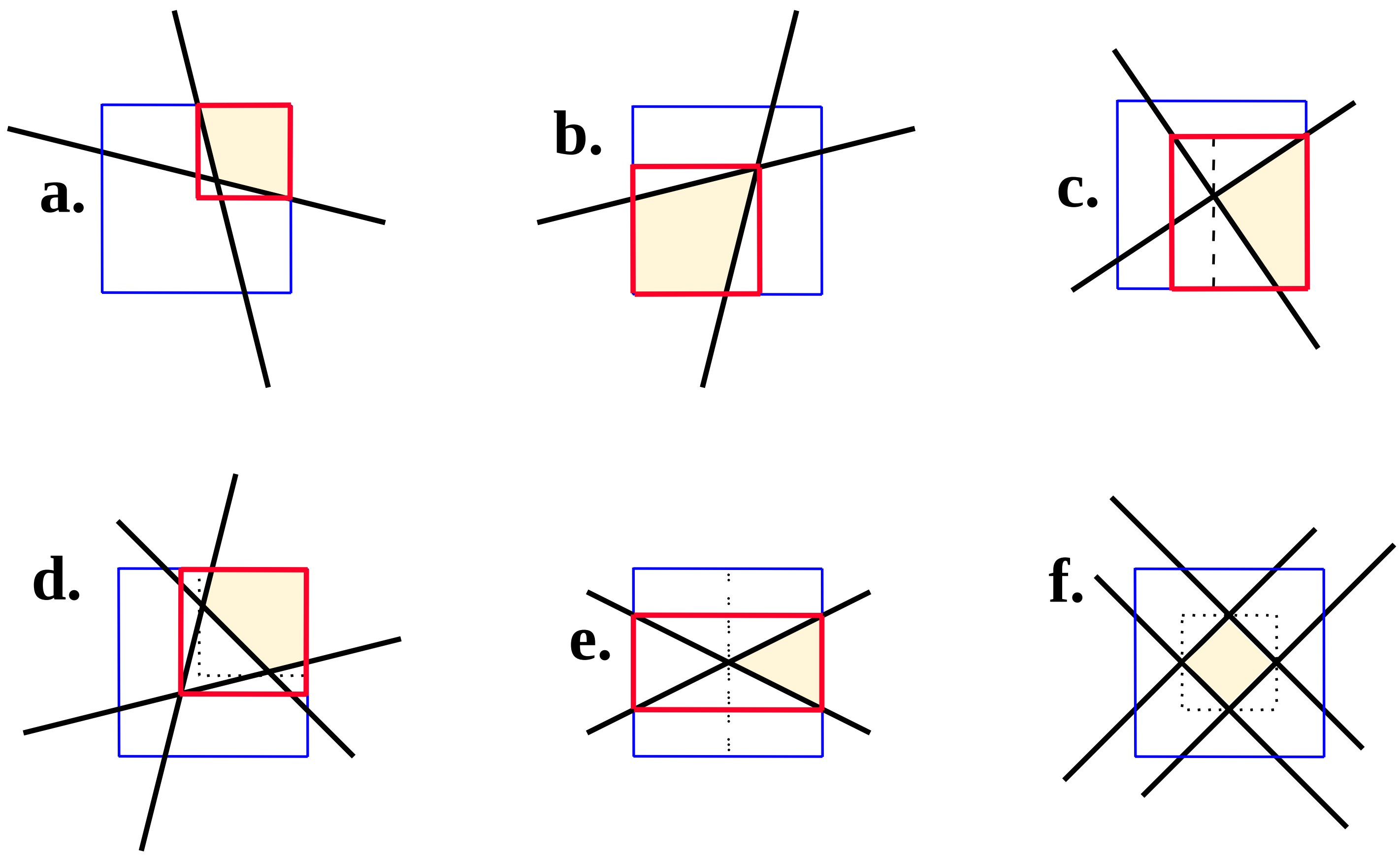
Q. What if a variable does not go to a bound?

A. Use probing: temporarily fix variable at a bound or minimize/maximize this variable over the problem constraints

Relaxed Value Function



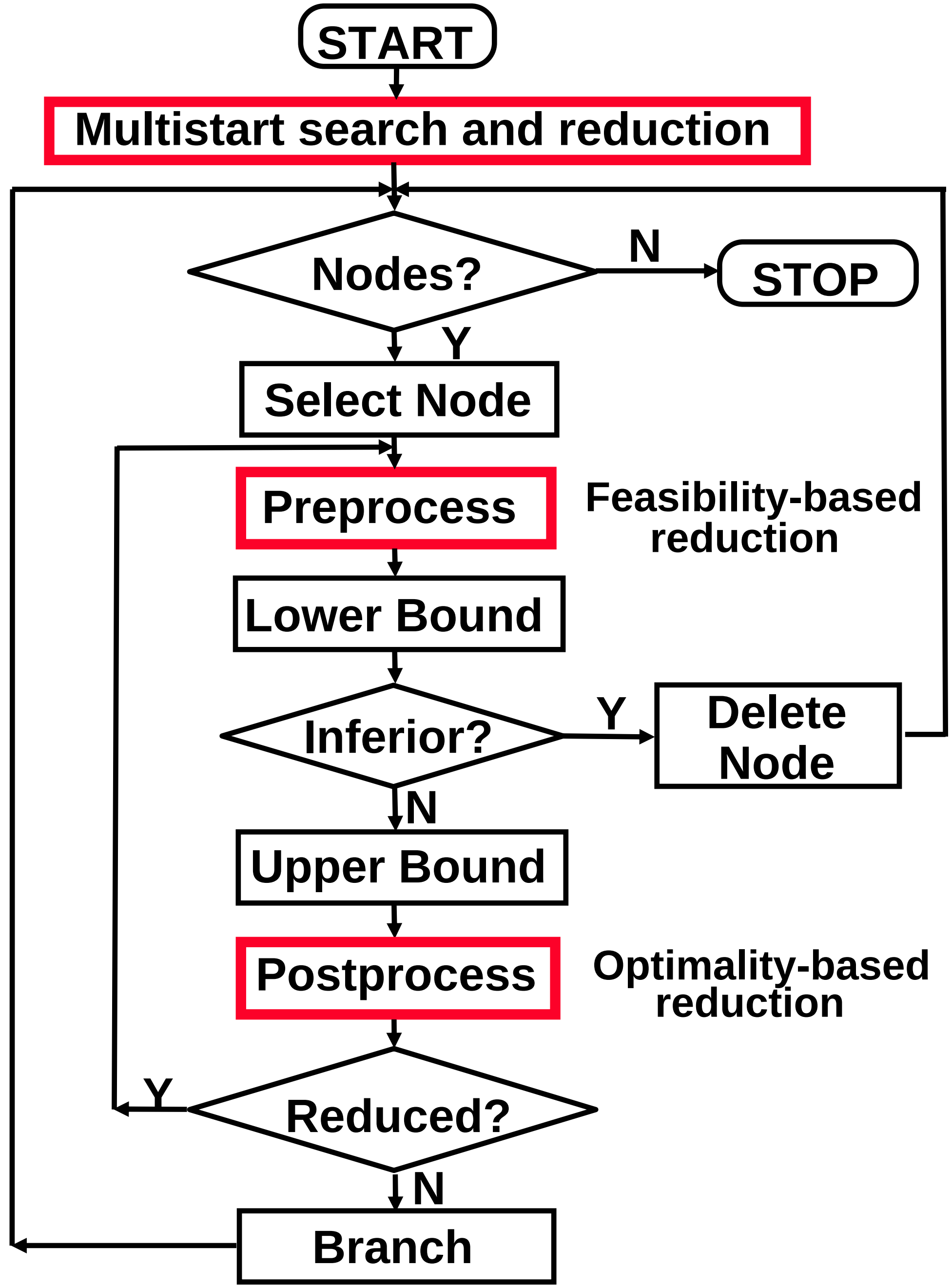
POOR MAN'S LPs AND NLPs



DOMAIN REDUCTION THEORY in T&S book (2002)

- Draw inferences about several reduction problems from solutions of problems solved in the course of branch-and-bound
- Implied results:
  - Monotone complementarity bounds for convex programs
    - Mangasarian and McLinden, 1985
  - Linearity-based tightening in integer programming
    - Andersen and Andersen, 1995
  - Marginals-based reduction
    - Ryoo and Sahinidis, 1995
  - Branch and Contract
    - Zamora and Grossmann, 1999
- New reduction strategies
  - Learning heuristic improves branching decisions

BRANCH-AND-REDUCE



TINY TEST PROBLEMS

Ex.	Cons.	Vars.	Source/In	Description
1	1	2	Sahinidis & Grossmann	bilinear constraint
2	3	3	Liebman et al. (GINO)	design of a water pumping system
3	7	10	Liebman et al. (GINO)	alkylation process optimization
4	1	3	Liebman et al. (GINO)	design of insulated tank
5	3	5	Liebman et al. (GINO)	heat exchanger network design
6	3	3	Liebman et al. (GINO)	chemical equilibrium
7	7	10	Liebman et al. (GINO)	pooling problem
8	2	2	Swaney	bilinear and quadratic constraints
9	1	2	Swaney	bilinear constraints and objective
10	1	2	Soland	nonlinear equality constraint
11	2	3	Westerberg & Shah	bilinearities, economies of scale
12	3	4	Stephanopoulos & Westerberg	design of two-stage process systems
13	3	2	Kocis & Grossmann	MINLP, process synthesis
14	10	7	Yuan et al.	MINLP, process synthesis
15	6	5	Kocis & Grossmann	MINLP, process synthesis
16	9	12	Floudas & Ciric	heat exchanger network synthesis
17	2	2	GINO	design of a reinforced concrete beam
18	4	2	Visweswaran & Floudas	quadratically constrained LP
19	2	2	Manousiouthakis & Sourlas	quadratically constrained QP
20	6	5	Manousiouthakis & Sourlas	reactor network design
21	6	5	Stephanopoulos & Westerberg	design of three-stage process system
22	5	2	Kalantari & Rosen	linearly constrained concave QP
23	2	2	Al-Khayyal & Falk	biconvex program
24	4	2	Thakur	linearly constrained concave QP
25	4	2	Falk & Soland	nonlinear fixed charge problem

STANDARD BRANCH-AND-BOUND

Ex.	N_{tot}	N_{opt}	N_{mem}	T
1	3	1	2	0.8
2	1007	1	200	210
3	2122*	1	113*	1245*
4	17	1	5	6.7
5	1000*	1	1000*	417*
6	1	1	1	0.3
7	205	1	37	43
8	43	1	8	1
9	2192*	1	1000*	330*
10	1	1	1	0.4
11	81	1	24	19
12	3	1	2	0.6
13	7	2	3	1.3
14	7	3	3	3.4
15	15	8	5	3.4
16	2323*	1	348*	1211*
17	1000*	1	1001*	166*
18	1	1	1	0.5
19	85	1	14	11.4
20	3162*	1	1001*	778*
21	7	1	4	1.2
22	9	1	4	1.2
23	75	6	13	11.7
24	7	3	2	1.5
25	17	9	9	2.9

*: Did not converge within limits of
$T \leq 1200$ (=20 min), and $N_{mem} \leq 1000$ nodes.

N_{tot} Total number of nodes
N_{opt} Node where optimum found
N_{mem} Max. no. nodes in memory
T CPU sec (SPARC 2)

- **Standard branch-and-bound converges very slowly**
- **It is not necessarily finite**
- **Tighter relaxations needed**

REDUCTION BENEFITS

Ex.	BRANCH-AND-BOUND N_{tot}	N_{opt}	N_{mem}	T	BRANCH-AND-REDUCE: No probing N_{tot}	N_{opt}	N_{mem}	T	BRANCH-AND-REDUCE: With Probing N_{tot}	N_{opt}	N_{mem}	T
1	3	1	2	0.8	1	1	1	0.5	1	1	1	0.7
2	1007	1	200	210	1	1	1	0.2	1	1	1	0.3
3	2122*	1	113*	1245*	31	1	7	20	9	1	5	48
4	17	1	5	6.7	3	1	2	0.4	1	1	1	0.3
5	1000*	1	1000*	417*	5	1	3	1.5	5	1	3	2.4
6	1	1	1	0.3	1	1	1	0.3	1	1	1	0.3
7	205	1	37	43	25	1	8	5.4	7	1	2	5.8
8	43	1	8	10	1	1	1	0.8	1	1	1	0.8
9	2192*	1	1000*	330*	19	1	8	5.4	13	1	4	7
10	1	1	1	0.4	1	1	1	0.4	1	1	1	0.4
11	81	1	24	19	3	1	2	0.6	1	1	1	0.7
12	3	1	2	0.6	1	1	1	0.2	1	1	1	0.2
13	7	2	3	1.3	3	1	2	0.7	1	1	1	0.7
14	7	3	3	3.4	7	3	3	2.7	3	3	2	3
15	15	8	5	3.4	1	1	1	0.3	1	1	1	0.3
16	2323*	1	348*	1211*	1	1	1	2.2	1	1	1	2.4
17	1000*	1	1001*	166*	1	1	1	3.7	1	1	1	4
18	1	1	1	0.5	1	1	1	0.5	1	1	1	0.6
19	85	1	14	11.4	9	1	4	1.8	1	1	1	1.4
20	3162*	1	1001*	778*	47	1	12	16.7	23	1	5	15.4
21	7	1	4	1.2	1	1	1	0.5	1	1	1	0.5
22	9	1	4	1.2	3	1	2	0.4	3	1	2	0.5
23	75	6	13	11.7	47	1	9	6.5	7	1	4	5
24	7	3	2	1.5	3	1	2	0.5	3	1	2	0.6
25	17	9	9	2.9	5	1	3	0.8	5	1	3	1

Branch-And-Reduce Optimization Navigator

Components

- Modeling language
- Preprocessor
- Data organizer
- I/O handler
- Range reduction
- Solver links
- Interval arithmetic
- Sparse matrix routines
- Automatic differentiator
- IEEE exception handler
- Debugging facilities

Capabilities

- Core module
  - Application-independent
  - Expandable
- Fully automated MINLP solver
- Application modules
  - Multiplicative programs
  - Indefinite QPs
  - Fixed-charge programs
  - Mixed-integer SDPs
  - …
- Solve relaxations using
  - CPLEX, XA, MINOS, SNOPT, OSL, SDPA, …

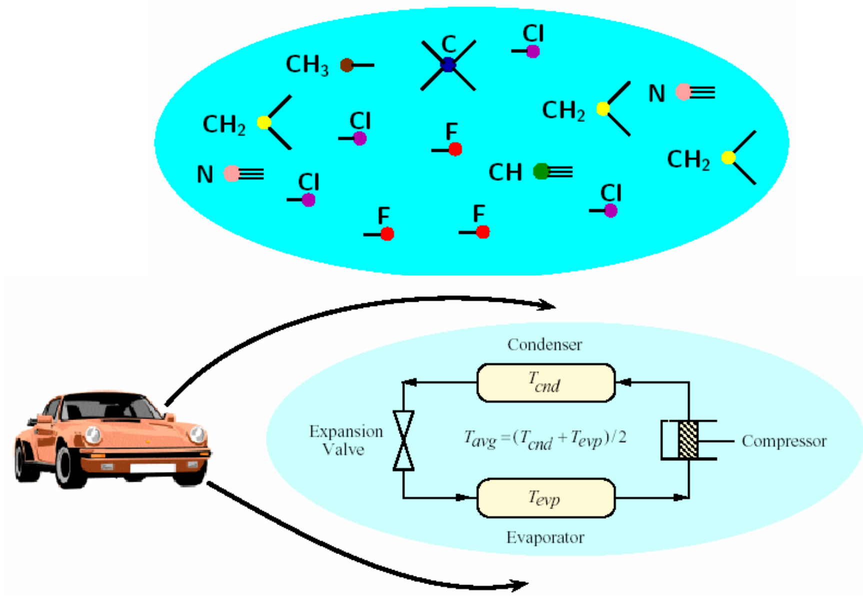
- First on the Internet in March 1995
- On-line solver between October 1999 and May 2003
  - Solved eight problems a day
- Available under GAMS and AIMMS

BARON IN APPLICATIONS

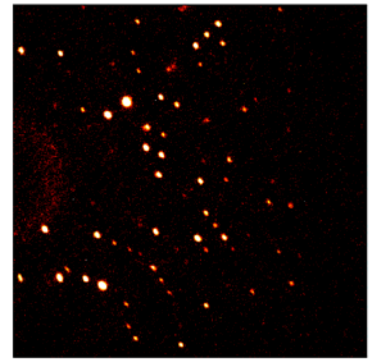
- **Development of new Runge-Kutta methods for partial differential equations**
  - **Ruuth and Spiteri, *SIAM J. Numerical Analysis,* 2004**
- **Energy policy making**
  - **Manne and Barreto, *Energy Economics,* 2004**
- **Design of metabolic pathways**
  - **Grossmann, Domach and others, *Computers & Chemical Engineering,* 2005**
- **Automatic control**
  - **Bemporand and Ljung, *Automatica,* 2004**
- **Product and process design**
  - **Koksalan and Plante, *Manufacturing and Service Operations Management,* 2003**
- **Design of containerships**
  - **Sherali and Ganesan, *Journal of Global Optimization,* 2003**

CHEM-, BIO- and MEDICAL INFORMATICS

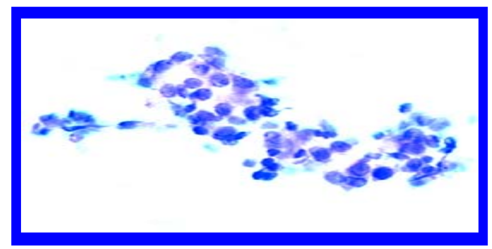
Molecular design



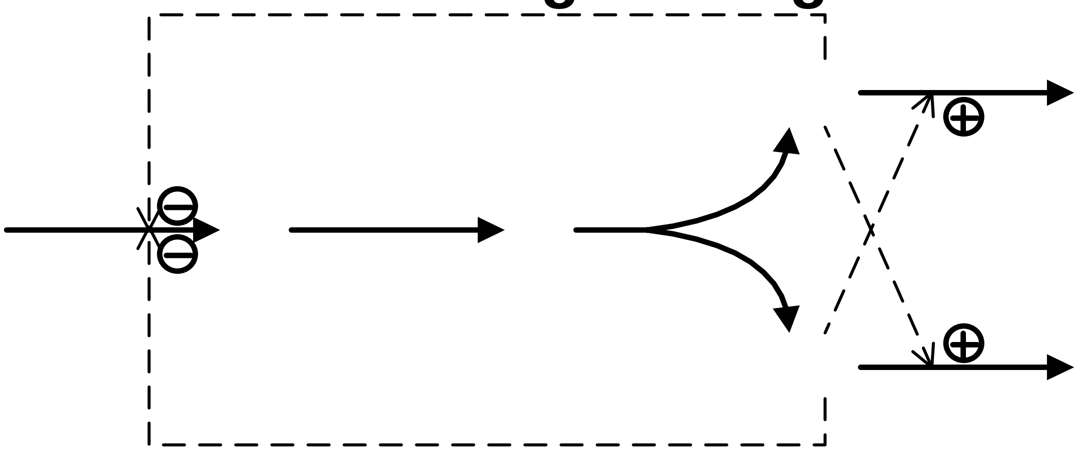
X-ray imaging



Breast cancer diagnosis



Metabolic engineering



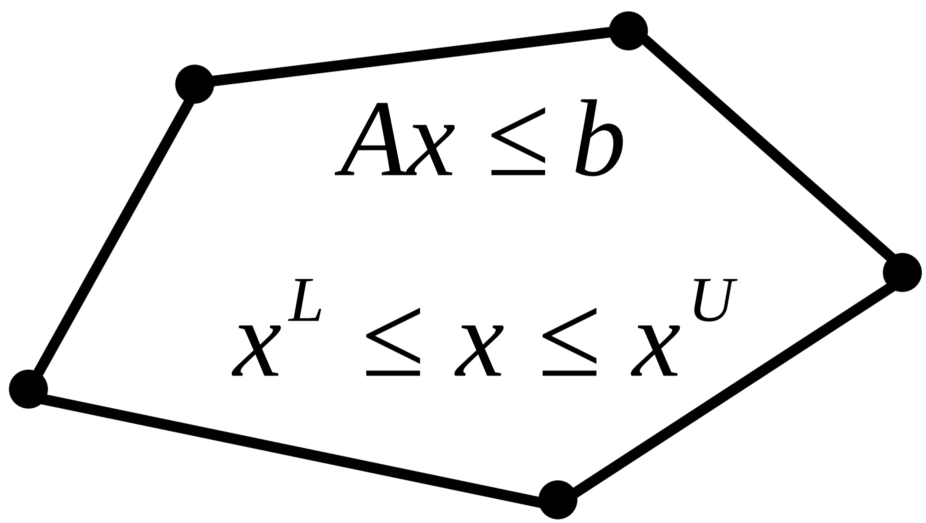
Bioinformatics

SEPARABLE CONCAVE MINIMIZATION

Consider $f_k(x_k)$ concave, $\forall k.$

$$\min\ f(x) = \sum_k f_k(x_k)$$

s.t.



PHILLIPS AND ROSEN PROBLEMS

			GOP (1993) E=1% (relative) HP 730		P&R (1990) E=0.1% (relative) CRAY 2 (parallel)			BARON (1996) ε=.000001 (absolute) IBM RS/6000 Power PC		
m	n	k	avg	std dev	min	avg	max	min	avg	max
20	25	0	0.5	0.01	1	2	4	0.3	0.4	0.5
20	25	50	2	2	1	1	1	1	1	1
20	25	100	17	20	1	2	3	1	2	3
20	25	200	33	28	2	7	17	2	4	6
20	25	400	82	58	7	14	32	4	10	16
20	50	0	0.6	0.01	3	6	13	1	1	1
20	50	50	17	31	1	2	3	2	2.5	3
20	50	100	47	49	2	5	14	2	4	7
20	50	200	109	80	4	9	28	4	8	19
20	50	400			20	32	45	11	20	48
40	25	0	0.5	0.02				0.3	0.4	0.4
40	25	50	1	0.6				1	1	1
40	25	100	3	4				1	2	3
40	25	200	25	26				2	4	5
40	25	400						6	15	22
50	100	0						6	7	14
50	100	50						8	12	18
50	100	100						9	17	27
50	100	200						14	65	160
50	100	400						131	345	663

- **(m, n/k) = number of constraints, concave/linear variables.**
- **HP 730 is 3-4 times faster than IBM RS/6000 Power PC.**
- **CRAY 2 is 10+ times faster than IBM RS/6000 Power PC.**

CUTTING PLANE GENERATION

- Use supporting hyperplanes (outer approximation) of convex functions from:
  - Univariate convex functions of original problem
  - Univariate convex functions obtained from functional decomposition of multivariate functions
  - Convex envelopes of nonconvex functions
  - Multivariate functions identified by CONVEX_EQUATIONS modeling language construct by the user
- Supporting hyperplanes generated only if they are violated by LP solution
- Process:
  - Start with a rough outer approximation
  - Solve LP
  - Add some cuts
  - Repeat process at current node

ILLUSTRATIVE EXAMPLE 1: CUTS FROM CONVEX ENVELOPES

$$\begin{aligned} \min \quad & x^2 - 100x + y^2 - 30y + 1000\frac{x}{y} \\ \text{s.t.} \quad & 0 \le x \le 1000 \\ & 1 \le y \le 1000 \end{aligned}$$

Solution -1118 at (34.3, 31.8)

Iteration	Lower bound	Relaxation optimal solution
1	-7500.9	$x_1 = (65.3, 66.5)$
2	-3832.2	$x_2 = (33.1, 34.1)$
3	-2839.5	$x_3 = (49.2, 19.0)$
4	-2325.7	$x_4 = (41.1, 25.6)$
5	-2057.5	$x_5 = (37.1, 22.3)$
6	-2041.1	$x_6 = (39.1, 23.9)$

Cutting planes reduce root-node gap by 86%

With cuts: 7 nodes
Without: 47 nodes

ILLUSTRATIVE EXAMPLE 2: CONVEX_EQUATIONS CONSTRUCT

$$\begin{aligned}
\min \quad & 100\log(e^{x_1}+e^{x_2}+e^{x_3})+x_1^2-40x_1 \\
& +x_2x_3-10\log(x_1)+x_2^2-20x_2-50x_3 \\
\text{s.t.} \quad & 1 \le x_1 \le 10 \\
& 1 \le x_2 \le 10 \\
& 1 \le x_3 \le 10
\end{aligned}$$

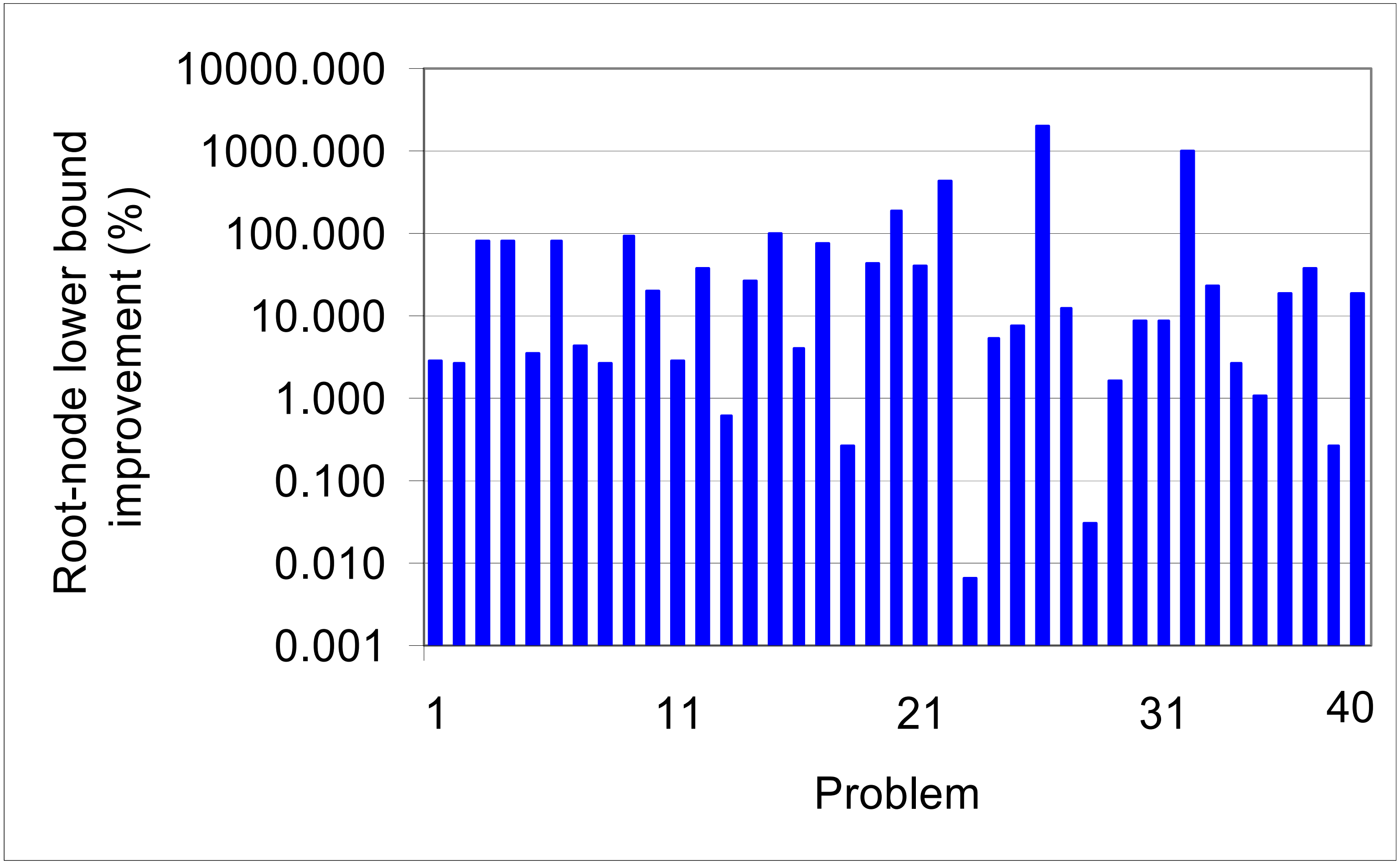
Solution 83.28

Iteration	Lower bound 1	Lower bound 2
1	−586.9	62.5
2	−514.9	78.9
3	−445.8	79.6
4	−436.2	80.2
5	−432.9	80.6

Cutting planes reduce root-node gap by 99.5%

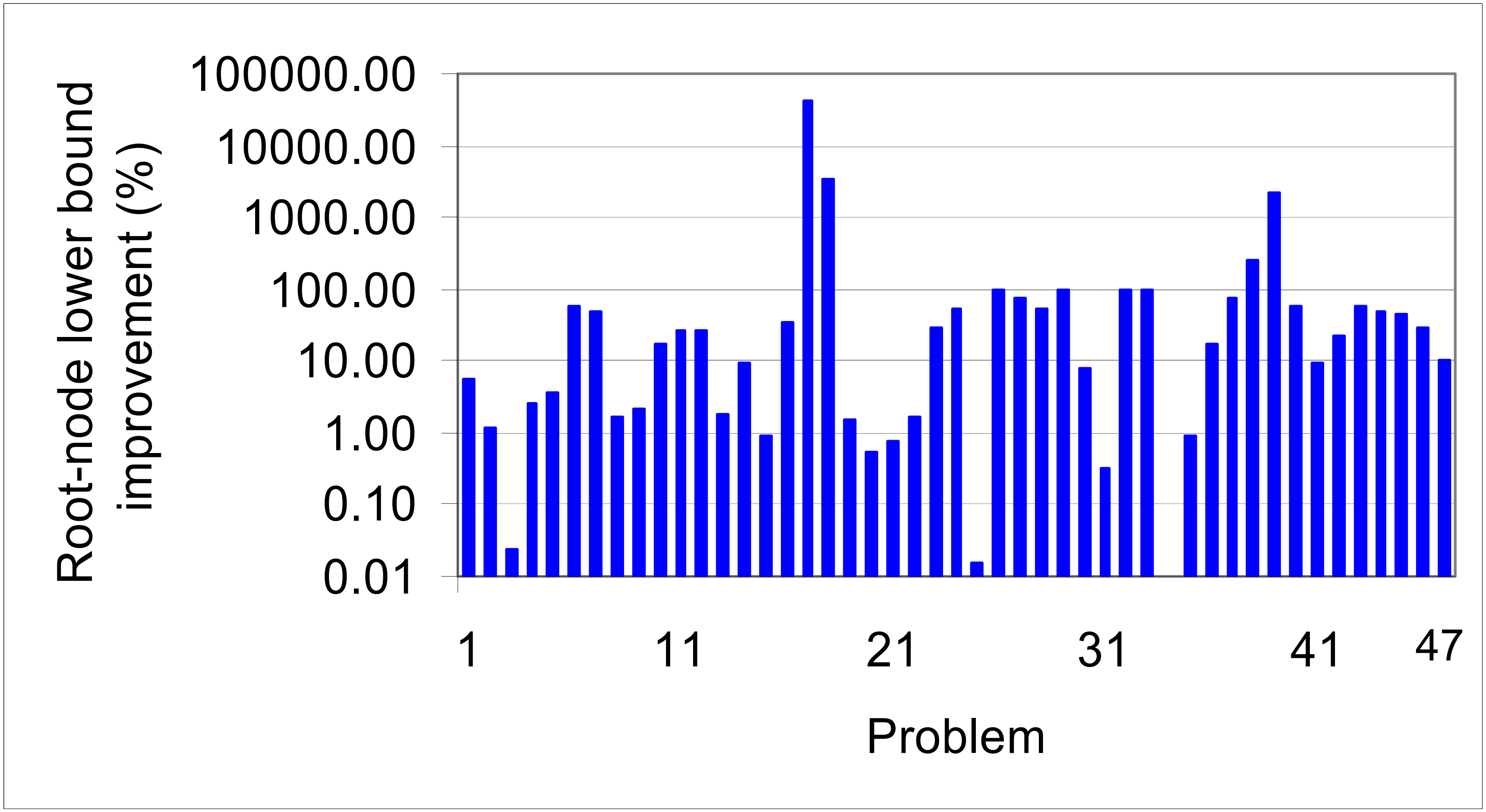
With cuts: 35 nodes
Without: 1793 nodes

ROOT-NODE LOWER BOUND IMPROVEMENTS FOR `globallib`



Up to 2000% improvement

ROOT-NODE LOWER BOUND IMPROVEMENTS FOR `minlplib`



Up to 41357% improvement

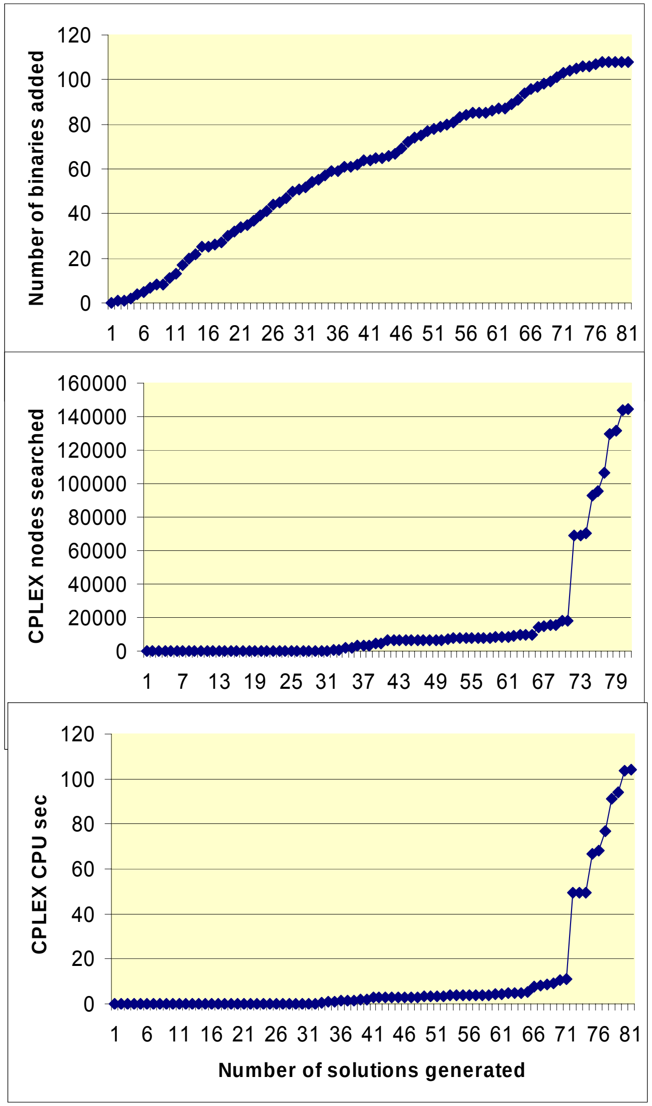
26 PROBLEMS FROM `globallib` AND `minlplib`

	Minimum	Maximum	Average
Constraints	2	513	76
Variables	4	1030	115
Discrete variables	0	432	63

EFFECT OF CUTTING PLANES

	Without cuts	With cuts	% reduction
Nodes	23,031,434	253,754	99
Nodes in memory	622,339	13,772	98
CPU sec	275,163	20,430	93

FINDING THE *K*-BEST OR ALL FEASIBLE SOLUTIONS



Typically found through repetitive applications of branch-and-bound and generation of "integer cuts"

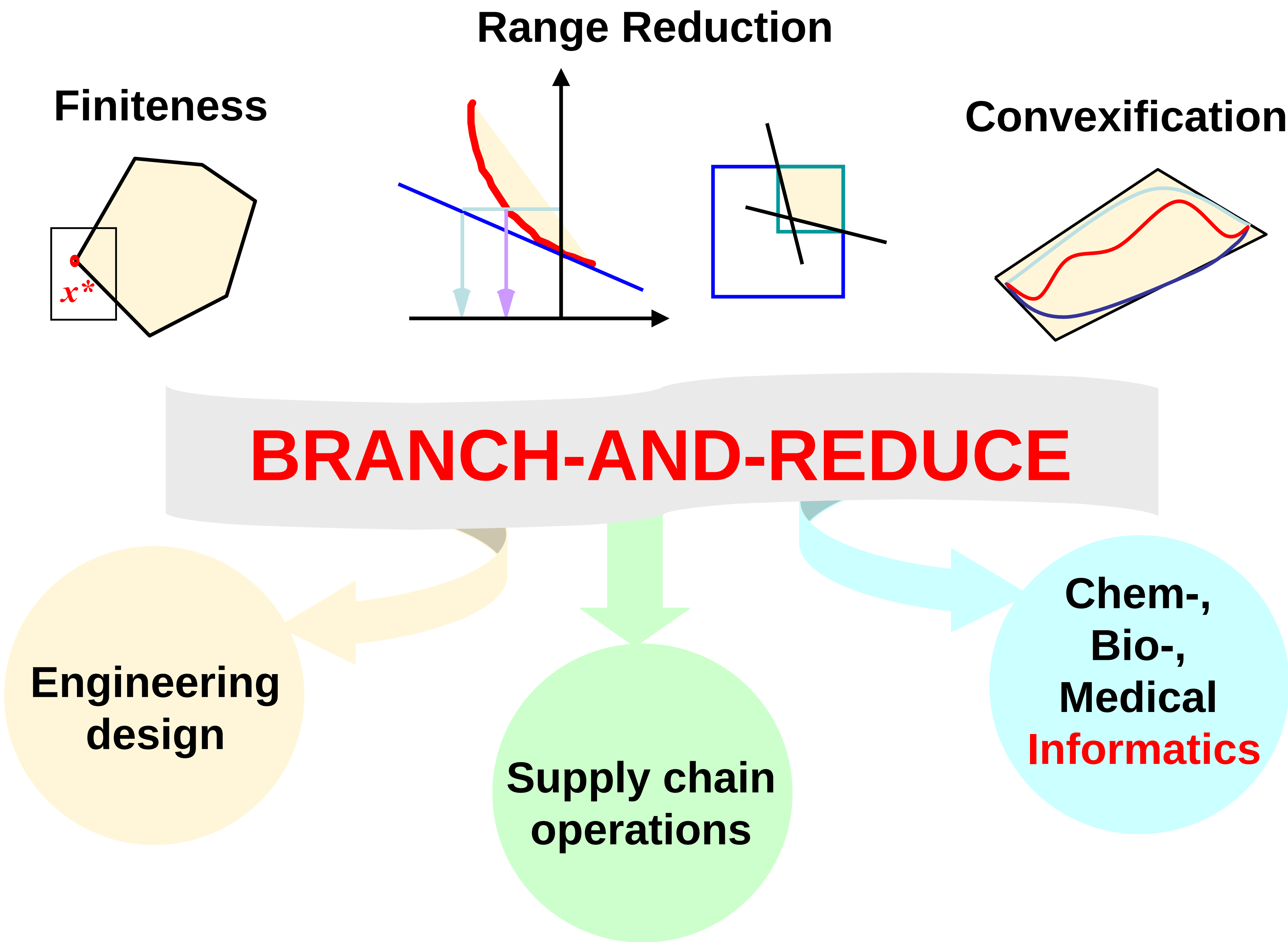
$$\min \quad \sum_{i=1}^{4} 10^{4-i} x_i$$

$$\text{s.t.} \quad 2 \le x_i \le 4, \quad i = 1,\ldots,4$$

$$x \quad \text{integer}$$

BARON finds all solutions:

- **No integer cuts**
- **Fathom nodes that are infeasible or points**
- **Single search tree**
- **511 nodes; 0.56 seconds**
- **Applicable to discrete and continuous spaces**


Range Reduction
Finiteness
Convexification
x*
BRANCH-AND-REDUCE
Engineering design
Supply chain operations
Chem-,
Bio-,
Medical
Informatics

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