

# VARIATIONAL INEQUALITIES OVER NORMAL REGIONS

---ANALYTIC CENTER CUTTING PLANE METHODS  
AND  
ECONOMIC EQUILIBRIUM PROBLEMS

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# **1. ECONOMIC EQUILIBRIUM PROBLEMS**

## 1) Walrasian equilibrium

**Model 1** Consider a perfect competition economy with  $n$  commodities.

The **aggregate excess demand function** is defined by

$$\begin{aligned} E(p) &= -D(p) + S(p) \\ &= (E_1(p), E_2(p), \dots, E_n(p)), \\ &\text{for price vector } p \in \mathbb{R}^n. \end{aligned}$$

Where  $D$ : demand function

$S$  : supply function

Traditionally,  $p^* \in \mathbf{S}$  is said to be a **Walrasian equilibrium price vector** if

$$E(p^*) \geq 0.$$

Where  $\mathbf{S} = \{x \in \mathbb{R}^n; x_j \geq 0, \sum_{j=1}^n x_j = 1\}$  is an  $(n - 1)$ -dimensional **unit simplex**.

It is known (A. Nagurney, 1993) that  $p^* \in \mathbb{R}^n$  is a **Walrasian equilibrium price vector if and only if** it is a solution of the **variational inequality**

$$E(p^*)^T (p - p^*) \geq 0, \forall p \in \mathbf{S}.$$

If each price of a commodity is involved in the **market structure which has a lower positive bound and may have an upper bound**, then the price are assumed to be contained in the **box-constrained set**

$$K = \{x \in \mathbb{R}^n; 0 \leq a_j \leq x_j \leq b_j \leq +\infty\}.$$

$p^* \in \mathbb{R}^n$  is a **Walrasian equilibrium price vector** if it is a solution of the variational inequality over a **box-constrained set (I. V. Konnov and E. O, Volotskaya 2002)**

$$E(p^*)^T (p - p^*) \geq 0, \forall p \in K.$$

A price vector  $p^* \in \mathbb{R}^n$  is a **Walrasian equilibrium price vector** if it is a solution to the **variational inequality**

$$E(p^*)^T (p - p^*) \geq 0, \forall p \in X.$$

Where  $X$  is a **normal region**.

The **convex hull** of a set  $B \subseteq \mathbb{R}^n$  is  
 $con(B) = \left\{ \sum_{i=1}^l \alpha_i x_i; x_i \in B, \sum_{i=1}^l \alpha_i = 1, \alpha_i \geq 0 \right\}$

A **polytope** is a set  $P \subseteq \mathbb{R}^n$  which is the **convex hull of a finite set**.

A **polyhedron** is a set

$$\{x \in \mathbb{R}^n; A^T x \leq b\} \subseteq \mathbb{R}^n,$$

where  $b \in \mathbb{R}^m$ , and  $A$  is an  $m \times n$  matrix ( $m \geq n$ ).

**Proposition 1**(H. Minkowski, 1896) A set  $P \subseteq \mathbb{R}^n$  is a polytope **if and only if** it is a bounded polyhedron.

**Definition 1** A bounded subset  $X \subseteq \mathbb{R}^n$  is said to be a **normal region** if there exists a sequence of polytopes  $\{C_j\}$ , which satisfies  $C_j \subseteq C_{j+1}$  ( $j = 1, \dots$ ), such that

$$\left(\bigcup_{j=1}^{\infty} C_j\right)^c = X.$$

**Theorem 1** A **convex** and **bounded closed region**  $X \subseteq \mathbb{R}^n$  is a normal region.

**Example 1** The function

$$A = P \frac{p_1(1 + p_1)^N}{(1 + p_1)^N - 1}$$

is known as the **mortgage payment amount** that should be paid periodically

for  $N$  periods

on a mortgage amount  $P$

at a periodic interest rate of  $p_1$

We may consider the mortgage as a type of commodity. The price of the mortgage would be the interest rate  $p_1$ --which means  $p_1$  dollars for each one hundred-mortgage for one period.

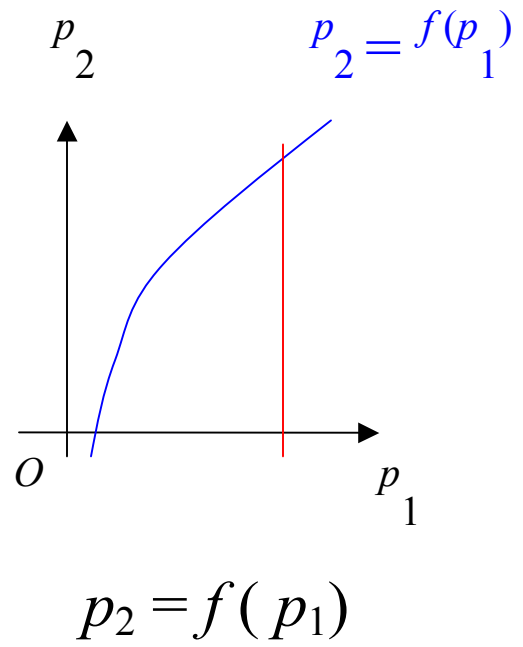
At the time of **the mortgage has been paid off**, the value  $p_2$  of the house satisfies

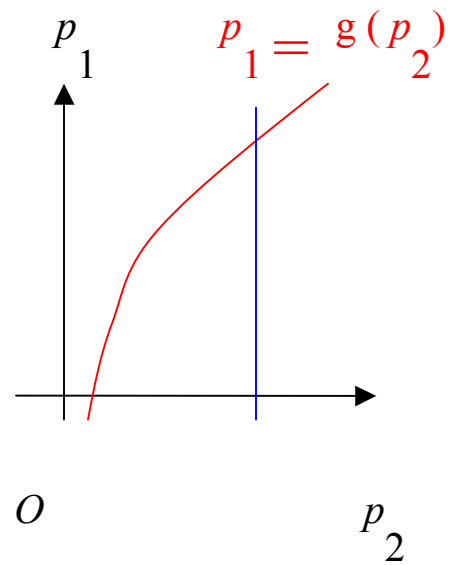
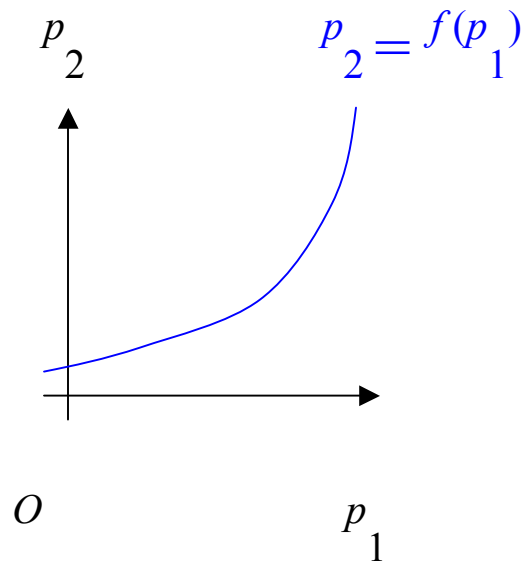
$$p_2 \geq B + A = B + P \frac{p_1(1+p_1)^N}{(1+p_1)^N - 1},$$

where  $B$  is the **down payment** in the beginning of the mortgage.

**Consider the function**

$$p_2 = f(p_1) = B + P \frac{p_1(1+p_1)^N}{(1+p_1)^N - 1}.$$





$$p_1 = g(p_2) = f^{-1}(p_2)$$

## 2) Nash equilibrium

**Model 2** Consider an **oligopolistic economy** in which a product is supplied by  $n$  firms.

Let  $p(\sigma)$  denote the **inverse demand function**, which is the price at which consumers will purchase a quantity  $\sigma$ . If each firm supplies  $q_i$  units of the product, then the **total supply** is

$$\sigma_q = \sum_{i=1}^n q_i.$$

We denote by  $h_i(q_i)$  the  $i$ th firm's total cost of supplying  $q_i$  units of the product, the **profit** of the  $i$ th firm is given by

$$\varphi_i(q) = q_i p(\sigma_q) - h_i(q_i).$$

A vector  $q^* = (q_1^*, q_2^*, \dots, q_n^*) \in \mathbb{R}^n$  is said to be a **Nash equilibrium solution** if it is an optimal solution to the problem

$$\max_{q \in K} \varphi_i(q) = \max_{q \in K} [q_i p(\sigma_q + \sigma_i^*) - h_i(q_i)].$$

Where  $K$  is the **box-constrained set**, and

$$\sigma_i^* = \sum_{j=1, j \neq i}^n q_j^* \quad (i = 1, 2, \dots, n).$$

Denote the partial derivative by  $\partial h_i$ , and  $G_i(q) = -p(\sigma_q) - q_i p'(\sigma_q)$ , ( $i = 1, 2, \dots, n$ ). Then, a vector  $q^*$  is a **Nash equilibrium solution** in the **oligopolistic economy** if and only if it is a solution to the problem

$$G(q^*)^T (q - q^*) + \sum_{i=1}^n \partial h_i(q_i^*)(q_i - q_i^*) \geq 0, \quad \forall q \in \mathbf{K}.$$

Where  $\mathbf{K}$  is the box-constraint set

$$\mathbf{K} = \{x \in \mathbf{R}^n; 0 \leq x_j \leq b_j\}.$$

**(I. V. Konnov, and E. O, Volotskaya 2002).**

Write  $H(q) = G(q) + \partial h(q)$ , then one has

$$H(q^*)^T (q - q^*) \geq 0, \forall q \in \mathbf{K}.$$

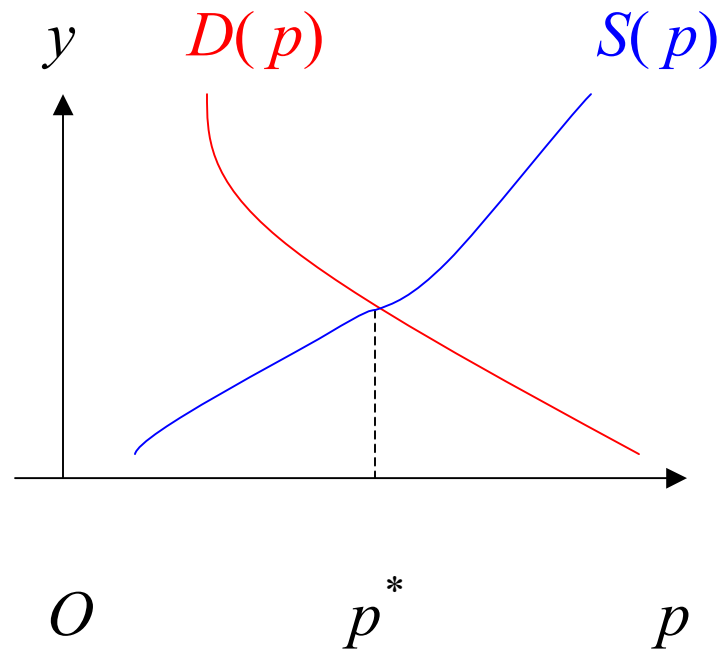
A **Nash equilibrium solution** in the **oligopolistic economy** can be defined as a solution of the problem:

$$H(q^*)^T (q - q^*) \geq 0, \forall q \in \mathbf{X}.$$

Where  $\mathbf{X}$  is a **normal region**.

A price vector  $p^* \in X$  is said to be a **stationary price vector** of the **Walrasian equilibrium problem** if there holds

$$E(p)^T (p - p^*) \geq 0, \forall p \in X.$$



Assume that  $E(p) = S(p) - D(p)$  is a **real valued** function.

$$1) E(p^*)^T (p - p^*) \geq 0, \forall p \in X \Leftrightarrow E(p^*) = 0 \\ \Leftrightarrow D(p^*) = S(p^*).$$

$$2) E(p^*)^T (p - p^*) \geq 0, \forall p \in X \\ \text{if } p \geq p^*, \forall p \in X \text{ then } E(p^*) \geq 0, \\ \text{i.e., } D(p^*) \leq S(p^*); \\ \text{if } p \leq p^*, \forall p \in X \text{ then } E(p^*) \leq 0, \\ \text{i.e., } D(p^*) \geq S(p^*).$$

A function  $F: X \rightarrow \mathbb{R}^n$  is **pseudomonotone** if

$$F(x)^T (y - x) \geq 0 \Rightarrow F(y)^T (y - x) \geq 0, \forall x, y \in X$$

If  $E$  is **continuous** and **pseudomonotone**, then  $p^* \in X$  is **Walrasian equilibrium price vector** **if and only if** it is a **stationary price vector**.

The function  $E: X \rightarrow \mathbb{R}^n$  is said to be **properly quasimonotone** (A. Daniilidis and N. Hadjisavvas, 1999) if  $\forall p_i \in X, i \in \{1, 2, \dots, m\}, \forall p \in \text{con}(p_1, p_2, \dots, p_m), \exists j \in \{1, 2, \dots, m\}$  such that

$$E(p_j)^T (p_j - p) \geq 0.$$

**Proposition 2** (R. John, 2001) There exists a stationary price vector  $p^*$  for the **Walrasian equilibrium problem** if and only if  $E: X \rightarrow \mathbb{R}^n$  is **properly quasimonotone**.

**Definition 2** The function  $E: X \rightarrow \mathbb{R}^n$  is said to be **strict-properly quasimonotone** if  $\forall p_i \in X, i \in \{1, 2, \dots, m\}, \forall p \in \text{con}(p_1, p_2, \dots, p_m)$  with  $p \neq p_i, i \in \{1, 2, \dots, m\}, \exists j \in \{1, 2, \dots, m\}$  such that

$$E(p_j)^T (p_j - p) > 0.$$

**Theorem 2** Let  $E: X \rightarrow \mathbb{R}^n$  be continuous. Then the following are equivalent:

(a)  $E: X \rightarrow \mathbb{R}^n$  is strict-properly quasimonotone;

(b) There exists a unique stationary price vector  $p^*$  for the economic equilibrium problem;

(c) There exists a unique Walrasian equilibrium price vector for the economic equilibrium problem.

## **2. ANALYTIC CENTER CUTTING PLANE METHODS**

# 1) Analytic Center Cutting Plane Algorithms for Pseudomonotone Variational Inequalities

A function  $F: X \rightarrow \mathbb{R}^n$  is **pseudomonotone**  
if

$$F(x)^T (y - x) \geq 0 \Rightarrow F(y)^T (y - x) \geq 0, \forall x, y \in X$$

Let  $X$  be a non-empty subset of  $\mathbb{R}^n$  and let  $F: X \rightarrow \mathbb{R}^n$  be a function. We call that a point  $x^* \in X$  is a solution of the **variational inequality**  $VI(F, X)$  if there holds

$$F(x^*)^T (x - x^*) \geq 0, \forall x \in X.$$

The point  $x^* \in X$  is a solution of the **dual variational inequality**  $VID(F, X)$  if there holds

$$F(x)^T (x - x^*) \geq 0, \forall x \in X.$$

We denote by  $X^*$  the set of solutions of  $VI(F, X)$ , and by  $X_D^*$  the set of solutions of  $VID(F, X)$ .

Define the **gap function**

$$g_X(x) = \max_{y \in X} F(x)^T (x - y), x \in X$$

A point  $x^* \in X$  is said to be a  **$\varepsilon$ -solution** of the variational inequality

$$F(x^*)^T (x - x^*) \geq 0, \forall x \in X$$

if

$$g_X(x^*) < \varepsilon.$$

For any **polytope**  $\{x \in \mathbf{R}^n; A^T x \leq b\}$ , Let  $\{x \in \mathbf{R}^n; A^T x + s = b, s = (s_1, s_2, \dots, s_n), s_i \geq 0\}$ .

**Potential function:**  $\varphi = \sum_{i=1}^n \ln s_i$ .

An **analytic center** is the maximizer of  $\varphi$ , and the unique solution of the system

$$A^T y = 0$$

$$A^T x + s = b$$

$$Y^T s = e,$$

where  $y$  is a positive dual vector, and  $Y$  the diagonal matrix built upon  $y$ .

An **approximate analytic center** is the maximizer of the potential function  $\varphi$ , and the unique solution of the system

$$A^T z = 0$$

$$A^T x + s = b$$

$$\| Z^T s - e \| \leq \eta < 1.$$

Where  $z$  is a dual vector, and  $Z$  the diagonal matrix built upon  $Z$ .

**Algorithm 1 (S1)**  $k = 0, j = 1, A^k = A_j, b^k = b_j,$

$$C_j^k = \{x \in \mathbf{R}^n; A_j^k x \leq b_j^k\};$$

**(S2)** Find an approximate analytic center  $x_j^k$  of  $C_j^k$

**(S3)** Compute  $g_X(x_j^k) = \max_{x \in X} F(x_j^k)^T (x_j^k - x)$

if  $g_X(x_j^k) = 0$ , **then** STOP, **else** GOTO S4

**(S4)** Compute  $g_{C_j}(x_j^k) = \max_{x \in C_j} F(x_j^k)^T (x_j^k - x)$

if  $g_{C_j}(x_j^k) < \frac{1}{2^j}$ , **then** increase  $j$  by 1 RETURN

TO Step1, **else** GOTO S5

**(S5)** Set  $A_j^{k+1} = \begin{bmatrix} A_j^k \\ F(x_j^k)^T \end{bmatrix}, b_j^{k+1} = \begin{bmatrix} b_j^k \\ F(x_j^k)^T x_j^k \end{bmatrix}$

$H_j^k = \{x \in \mathbf{R}^n; F(x_j^k)^T (x - x_j^k) = 0\}$  is the new cutting plane for  $VI(F, C_j^k)$ . Increase  $k$  by one GOTO S2.

**Algorithm 1** modifies P. J. L. Goffin, P. Marcotte, and D. L. Zhu's (1997) algorithm.

A function  $F: X \rightarrow \mathbb{R}^n$  is **pseudomonotone plus** on  $X$  if

$$\begin{aligned} F(x)^T (y - x) \geq 0 \\ F(y)^T (y - x) = 0 \end{aligned} \Rightarrow F(x) = F(y), \forall x, y \in X.$$

**Theorem 3** Let  $F: X \rightarrow \mathbb{R}^n$  be **pseudomonotone plus** on a normal region  $X$ , then Algorithm 1 **either stops with a solution** of  $VI(F, X)$  after a finite number of iterations, **or there exists a subsequence** of the infinite sequence  $\{x_j^k\}$  that **converges** to a point  $x^* \in X^*$ .

**Algorithm 2** S1, S2, S4, and S5 are same as those of Algorithm 1.

**(S3)** Compute

$$g_X(x_j^k) = \max_{x \in X} F(x_j^k)^T (x_j^k - x),$$

if  $g_X(x_j^k) < \varepsilon$ , **then** STOP,

**else** GOTO S4.

**Theorem 4** Let  $F: X \rightarrow \mathbb{R}^n$  be pseudomonotone plus on a normal region  $X$ , then Algorithm 2 stops with a  $\varepsilon$ -solution of  $VI(F, X)$  after a finite number of iterations.

## 2) Analytic Center Cutting Plane Algorithms for Quasimonotone Variational Inequalities

A function  $F: X \rightarrow \mathbb{R}^n$  is **quasimonotone** if  
 $F(x)^T (y - x) > 0 \Rightarrow F(y)^T (y - x) \geq 0, \forall x, y \in X$

The following **Algorithm 3** modifies **P. Marcotte and D. L. Zhu's (2001)** approach.

For any given  $j$ , let

$$\Gamma_j(y, x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

be continuous in  $x$  and strong monotone in  $y$ ,  
i.e.,

$$\langle \Gamma_j(y', x) - \Gamma_j(y'', x), y' - y'' \rangle \geq \beta_j \|y' - y''\|^2, \\ \forall y', y'' \in X$$

for  $\beta_j > 0$ .  $\beta_j$  are said to be the strong  
monotonicity constant. The function  $\Gamma_j$  is  
associated with the variational inequality  
 $AVI[\Gamma, X, x]$  whose solution  $w_j(x)$  satisfies

$$\langle \Gamma_j(w_j(x), x) - \Gamma_j(x, x) + F(x), y - w_j(x) \rangle \geq 0 \\ \forall y \in X$$

$w_j(x)$  are continuous, and that  $x$  is a solution of  $VI[F, C_j]$  if and only if it is a fixed point of  $w$ .

Let  $\rho_j$  and  $\alpha_j$  be positive numbers less than 1 and  $\beta_j$ , respectively. Let  $l(j)$  (which depends on  $x$ ) be the smallest nonnegative integer for which

$$\begin{aligned} \langle F(x + \rho_j^{l(j)} (w_j(x) - x)), x - w_j(x) \rangle \\ \geq \alpha_j \| w_j(x) - x_j \|^2 \end{aligned}$$

Define

$$G_j(x) = F(x + \rho_j^{l(j)} (w_j(x) - x)).$$

If  $x_j^*$  is a solution of  $VI[F, C_j]$ , then

$$w_j(x_j^*) = x_j^*, l(j) = 0, \text{ and } G_j(x_j^*) = F(x_j^*).$$

**Algorithm 3 (S1)** Let  $\beta_j > 0$  be the strong  
monotonicity constant for  
 $\Gamma_j(y, x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with respect to  $y$  and let  
 $\alpha_j \in (0, \beta_j)$ .

$$k = 0, j = 1, A^k = A_j, b^k = b_j,$$

$$C_j^k = \{x \in \mathbb{R}^n; A_j^k x \leq b_j^k\}$$

**(S2)** Find an approximate analytic center  
 $x_j^k$  of  $C_j^k$

**(S3)** Compute  $g_X(x_j^k) = \max_{x \in X} F(x_j^k)^T (x_j^k - x)$

if  $g_X(x_j^k) = 0$ , **then STOP, else GOTO S4**

**(S4)** Compute  $g_{C_j}(x_j^k) = \max_{x \in C_j} F(x_j^k)^T (x_j^k - x)$

if  $g_{C_j}(x_j^k) < \frac{1}{j}$ , **then increase  $j$  by 1**

**RETURN TO S1, else GOTO S5**

**(S5)** Let  $w_j(x_j^k)$  satisfies

$$\begin{aligned} \langle F(x_j^k) + \Gamma_j(w_j(x_j^k), x_j^k) - \Gamma_j(x_j^k, x_j^k), y - w_j(x_j^k) \rangle \\ \geq 0, \forall y \in C_j \end{aligned}$$

$$y_j^k = x_j^k + \rho_j^{l(k,j)}(w_j(x_j^k) - x_j^k)$$

$$G_j(x_j^k) = F(y_j^k),$$

where  $l(k, j)$  is the smallest integer which satisfies

$$\begin{aligned} \langle F(x_j^k + \rho_j^{l(k,j)}(w_j(x_j^k) - x_j^k)), x_j^k - w_j(x_j^k) \rangle \\ \geq \alpha_j \|w_j(x_j^k) - x_j^k\|^2 \end{aligned}$$

**(S6)** Set

$$A_j^{k+1} = \begin{bmatrix} A_j^k \\ G(x_j^k)^T \end{bmatrix}, \quad b_j^{k+1} = \begin{bmatrix} b_j^k \\ G(x_j^k)^T x_j^k \end{bmatrix},$$

Let  $H_j^k = \{x \in \mathbb{R}^n; G(x_j^k)^T (x - x_j^k) = 0\}$ ,

increase  $k$  by one GOTO **S2**.

**Theorem 5** Let  $F : X \rightarrow \mathbb{R}^n$  be Lipschitz continuous on a normal region  $X$  and  $X_D^*$  be nonempty. Then Algorithm 3 **either** stops with a solution of  $VI(F, X)$  after a finite number of iterations, **or** there exists a subsequence of the infinite sequence  $\{x_j^k\}$  that converges to a point  $x^* \in X^*$ .

**Algorithm 4** S1, S2, S4, S5, and S6 are same as those in Algorithm 3.

(S3) Compute

$$g_X(x_j^k) = \max_{x \in X} F(x_j^k)^T (x_j^k - x),$$

if  $g_X(x_j^k) < \varepsilon$ , **then STOP**,

**else GOTO S4.**

**Theorem 6** Let  $F : X \rightarrow \mathbb{R}^n$  be Lipschitz continuous on a normal region  $X$  and  $X_D^*$  be nonempty. Then **Algorithm 4 stops** with a  $\varepsilon$ -solution of  $VI(F, X)$  **after a finite number of iterations.**

We should mention that our approach can be used to extend many interior point methods for optimization problems over polyhedral feasible sets, e.g., the algorithms given by

**J. E. Mitchell and S. Ramaswamy**, A Long-Step, Cutting Plane Algorithm for Linear and Convex Programming, *Annals of Operational Research*, 99 (2000): 95-122.

**J. L. Goffin and J. P. Vial**, Multiple Cuts in the Analytic Center Cutting Plane Method, *SIAM Journal on Optimization*, 11(2000)1: 266-288.

We note that we may apply our idea to other situations such as traffic equilibrium problems, electrical equilibrium problems, or non-cooperative games.