Pyramidal traveling salesman problem

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Scope and purpose

The problem of identifying polynomially testable and polynomially solvable subclasses of the TSP is of significant theoretical and practical value. Pyramidal TSP is a well-studied subclass of the TSP that is polynomially solvable. However, it is an NP-hard problem to test if a given instance of TSP is pyramidal. Various polynomially testable subclasses of this have been identified in literature. In this paper, we give a general polynomially testable sufficiency condition for an instance of TSP to be pyramidal. This provides simpler proofs and also a proper generalization of the known classes.

Abstract

In this paper, we give new polynomially testable sufficiency conditions for a given instance of the traveling salesman problem (TSP) to have an optimal tour that is pyramidal. This properly generalizes the Demidenko condition and the conditions of Warren. We thus have new, nontrivial polynomially testable and polynomially solvable cases of TSP. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Given a complete, directed graph $G$ on node set $N \equiv \{1, 2, \ldots, n\}$ and an $n \times n$ cost matrix, $C$, such that the cost of traversing arc $(i, j)$ is $C(i, j)$ (we assume throughout that $C(i, i) = 0 \ \forall i$), the traveling salesman problem (TSP) is to find a tour $\tau$ (see the definition in Section 2) such that $C(\tau) (= \sum C(i, j); (i, j) \in \tau)$ is minimum.

Since the TSP is NP-hard (see [4]), researchers have identified several polynomially testable and polynomially solvable special cases of it (see [1, 5]). Among the well-studied solvable classes is the one for which there exists an optimal tour which is pyramidal (i.e., of the form $(1 - i_1 - i_2 \cdots - i_r - n - j_1 \cdots - j_k - 1)$, where $i_1 < i_2 < \cdots < i_r$ and $j_1 > j_2 > \cdots > j_k$). Significance of this case lies in the fact that for a given ordering of the nodes, (i) the total number of pyramidal tours is exponentially large and (ii) an optimal pyramidal tour can be computed in $O(n^2)$ time (see [5]). Testing whether a given instance of TSP is pyramidal solvable is, however, an NP-hard problem.

During the last three decades, there has been a great deal of work, specially in the Russian literature, on identification of polynomially testable sufficiency conditions for an instance of TSP to be pyramidal solvable. We refer the reader to [1, 5] for further details on this. As shown in [1], the most general conditions known to date for a general (not necessarily symmetric) cost matrix, $C$, are (i) the Demidenko condition [3], (ii) the four sets of conditions by Baki and Kabadi [1, 2] and (iii) the eight sets of conditions by Warren [8]. In addition, Van Der Veen has given a set of sufficiency conditions for symmetric cost matrices [7]. (All these conditions are given in Appendix A.)

In this paper we introduce the concept of hereditary property of square matrices. This enables us to obtain a new polynomially testable sufficiency condition for an instance of TSP to be pyramidal solvable. We show that this new sufficiency condition properly generalizes the Demidenko condition and the Warren conditions. We thus have new polynomially testable and polynomially solvable cases of TSP.

In Section 2 we introduce notations, basic definitions and some basic results. In Sections 3 and 4, we present proper generalizations of the Demidenko condition. In Section 5, we give a new class of sufficiency conditions which properly generalize Warren conditions. Finally, in Section 6 we present a common generalization of all the results in the previous sections.

2. Notation, definitions and basic results

A path, $\mu$, from a node $u$ to a node $v$, denoted by $[u, v]$, is a sequence of arcs of the type $(k_0, k_1) - (k_1, k_2) - \cdots - (k_m, k_{m+1})$ where $k_0 = u$ and $k_{m+1} = v$. We shall refer to such a path by $(k_0 - k_1 - \cdots - k_m - k_{m+1})$. In this case, $\mu^{-1}$ is the path $(k_{m+1} - k_m - \cdots - k_1 - k_0)$. For any $i < m + 1$ and any $j > 0$, we define $\mu(k_i) = k_{i+1}$ and $\mu^{-1}(k_j) = k_{j-1}$ and, node $k_{i+1}$ ($k_{j-1}$) is said to be adjacent to node $k_i$ ($k_j$) in $\mu$. For any $0 \leq r < s \leq m + 1$, $(k_r - k_{r+1} - \cdots - k_s)$ is a subpath of $\mu$. The path $\mu$ is simple if all the nodes $\{k_i: i = 0, 1, \ldots, m + 1\}$ are distinct, except possibly the end nodes. If $u = v$, then the path $\mu$ is called closed and in this case, $\mu(u) = k_1$ and $\mu^{-1}(u) = k_m$ (i.e. nodes $k_1$ and $k_m$ are adjacent to node $u$). A tour is a simple, closed path covering all the nodes in $N$.

For any tour $\tau$, its reverse tour $\tau_R$ is given by $\tau_R(n - i + 1) = n - \tau(i) + 1 \ \forall 1 \leq i \leq n$. 
A peak of a path, $\mu$ is a node $p$ in $\mu$ such that (i) $p > \mu(p)$ (if $\mu(p)$ exists) and (ii) $p > \mu^{-1}(p)$ (if $\mu^{-1}(p)$ exists) and a valley of $\mu$ is a node $q$ in $\mu$ such that (i) $q < \mu(q)$ (if $\mu(q)$ exists) and (ii) $q < \mu^{-1}(q)$ (if $\mu^{-1}(q)$ exists). A pyramidal path is a path with only one peak and/or only one valley.

For an $n \times n$ cost matrix, $C$, $C^T$ is the transpose of $C$ and $C_R$ is the reverse of $C$. i.e. $\forall 1 \leq i, j \leq n$, $C_R(i, j) = C(n - i + 1, n - j + 1)$. The cost $C(\mu)$ of a path $\mu$ corresponding to cost matrix $C$ is the sum of costs of the arcs in $\mu$.

**Definition 1.** An instance of TSP, with a given numbering of nodes, is said to be pyramidal solvable if there exists an optimal tour that is pyramidal.

**Fact 1.** Checking if a given instance of TSP, with a given node numbering, is pyramidal solvable is NP-complete even if the cost matrix is symmetric and obeys the triangle inequality.

**Observation 1.** For any tour $\tau$, $\tau$ and $\tau^{-1}$ have the same sets of peaks and valleys. Node $i$ is a peak (valley) of $\tau$ iff node $(n - i + 1)$ is a valley (peak) of $\tau_R$. If $\tau$ is an optimal tour corresponding to cost matrix $C$, then $\tau^{-1}$ is an optimal tour corresponding to the cost matrix $C^T$ and $\tau_R$ is an optimal tour corresponding to the cost matrix $C_R$. Hence, there exists an optimal pyramidal tour corresponding to a cost matrix $C$ iff there exist optimal pyramidal tours corresponding to each of the cost matrices $C^T$, $C_R$ and $C_R^T$.

**Definition 2.** For an $n \times n$ cost matrix $C$, and for any $k < n$;

(i) $C^{(k)}$ is the submatrix of $C$ obtained by deleting all the rows and columns of $C$ indexed by $\{k + 1, k + 2, \ldots, n\}$.

(ii) $C^{(i,j,k)}$ is a $(k + 1) \times (k + 1)$ submatrix of $C$ obtained by deleting all the rows and columns of $C$ indexed $\{k + 1, k + 2, \ldots, n\}$ except row $i$ and column $j$ for $k < i, j \leq n, i \neq j$. We shall call $C^{(i,j,k)}$ an upper minor of $C$.

(iii) $C^{(k)}$ is the submatrix of $C$ obtained by deleting all the rows and columns of $C$ indexed by $\{1, 2, \ldots, k - 1\}$.

(iv) $C^{(i,j,k)}$ is a $(n - k + 2) \times (n - k + 2)$ submatrix of $C$ obtained by deleting all the rows and columns of $C$ indexed by $\{1, 2, \ldots, k - 1\}$, except row $i$ and column $j$ for $i, j < k, i \neq j$. We shall call $C^{(i,j,k)}$ a lower minor of $C$.

**Definition 3.** (i) A property $P$ of square matrices is upper hereditary if for any cost matrix $C$ satisfying property $P$, every upper minor of $C$ satisfies property $P$.

(ii) A property $P$ of square matrices is lower hereditary if for any cost matrix $C$, satisfying property $P$, every lower minor of $C$ satisfies property $P$.

(iii) A property $P$ of square matrices is hereditary if it is both lower hereditary and upper hereditary.

**Definition 4.** For any matrix property, $P$, $\Delta(P) = \min\{\Delta | \exists$ a $\Delta(P) \times \Delta(P)$ matrix which satisfies property $P$ and for which none of the optimal tours is pyramidal (obviously, $\Delta(P) \geq 4$).
Lemma 1. Let \( P \) be an upper hereditary (lower hereditary) matrix property with \( \Delta(P) = n \) for some \( n < \infty \). Then, for any \( n \times n \) cost matrix, \( C \), satisfying property \( P \) and for which none of the optimal tours is pyramidal, every optimal tour has node \((n - 1)\) as one of its peaks (node 2 as one of its valleys).

Proof. Let \( P \) be an upper hereditary matrix property with \( \Delta(P) = n \) for some \( n < \infty \). (The case when \( P \) is a lower hereditary matrix property will follow similarly.) If possible, let \( C \) be an \( n \times n \) cost matrix satisfying property \( P \) for which none of the optimal tours is pyramidal and for which there exists an optimal tour \( \tau \) with the second largest peak \( p < n - 1 \). Then, there exists a pyramidal subpath \( \mu = [i, j] \) of \( \tau \) with node set precisely \( \{p + 1, \ldots, n\} \). Replace the subpath \( \mu \) in \( \tau \) by a new node, \( \beta \), to get a tour \( \tau' \) on node set \( \{1, 2, \ldots, p, \beta\} \). \( C^{j,i,p} \) is an upper minor of \( C \) and \( P \) is an upper hereditary property. Hence, by the choice of \( n \), there exists an optimal pyramidal tour \( \tau'' \) on \( \{1, 2, \ldots, \beta\} \) corresponding to cost matrix \( C' = C^{j,i,p} \) with node \( \beta \) as the \((p + 1)\)th node. Replace the node \( \beta \) in \( \tau'' \) by the path \( \mu \) to get a pyramidal tour \( \tau''' \) on \( \{1, 2, \ldots, n\} \). Then,

\[
C(\tau''') - C(\tau) = C'(\tau') - C'(\tau) \leq 0
\]

We thus have a contradiction to the choice of matrix \( C \). This proves the lemma. \( \square \)

For any \( n \times n \) matrix \( A \) and any \( 1 \leq i \leq k \leq n \), and \( 1 \leq j \leq l \leq n \), let

\[
R_A(i, j, k, l) = A(i, l) + A(k, j) - A(i, j) - A(k, l).
\]

For convenience, we shall omit the suffix \( A \) wherever there is no ambiguity.

Definition 5. For any tour \( \tau \) on node set \( \{1, 2, \ldots, n\} \) and any \( i \in \{2, 3, \ldots, n\} \), \( \tau^{(i)} \) is a tour on node set \( \{1, 2, \ldots, i\} \) defined recursively as

\[
\tau^{(i)} = \tau.
\]

For \( i = n - 1, n - 2, \ldots, 2 \); \( \tau^{(i)} \) is obtained from \( \tau^{(i+1)} \) by replacing subpath \((u - (i + 1) - v)\) of \( \tau^{(i+1)} \) by arc \((u, v)\).

Let

\[
C^{\tau,i} = C^{(i)}(\tau^{(i)}) - C^{(i-1)}(\tau^{(i-1)}).
\]

It is easy to see that

\[
C^{\tau,i} = R(u, v, i, i) \text{ where, } u, v < i \text{ are distinct nodes.}
\]

As an example, let us consider the following tour, \( \tau \), on node set \{1, 2, 3, 4, 5, 6\}:

\[
\tau \equiv (1-5-3-6-2-4).
\]

Then,

\[
\tau^{(6)} = \tau, \quad \tau^{(5)} = (1-5-3-2-4), \quad \tau^{(4)} = (1-3-2-4), \quad \tau^{(3)} = (1-3-2), \quad \tau^{(2)} = (1-2);
\]

\[
C^{\tau,6} = R(3,2,6,6), \quad C^{\tau,5} = R(1,3,5,5), \quad C^{\tau,4} = R(2,1,4,4), \quad C^{\tau,3} = R(1,2,3,3).
\]
In the next four sections, we shall present new sufficiency conditions for existence of a pyramidal optimal tour, generalizing the Demidenko condition [3] and the Warren conditions [8].

3. A generalization of Demidenko condition

**Definition 6.** An \( n \times n \) cost matrix \( A \) satisfies property \( \mathbf{P}_1 \) iff \( A \) and \( A^T \) satisfy the following:

(i) For any \( \{i, j, x\} \) such that \( 2 < i < j < x \leq n \), and any distinct \( u, v, k \leq k \) for each \( k \in \{i, i + 1, \ldots, j\} \),

\[
\sum_{k=i}^{j} R(k, k-1, x, k) + \sum_{k=i}^{j} R(u, v, k, k) \geq 0.
\]

(ii) For any \( \{i, x\} \) such that \( 2 < i < x \leq n \) and any distinct \( u, v < i \), such that \( v < (i-1) \),

\[
R(i, i-1, x, i) + R(u, v, i, i) \geq 0.
\]

**Theorem 1.** If cost matrix \( C \) satisfies property \( \mathbf{P}_1 \), then there exists an optimal tour that is pyramidal.

**Proof.** From the fact that in the definition of property \( \mathbf{P}_1 \), we require \( 2 < i < j < n \), and \( k \leq j \), and that any two consecutive rows/columns (except the first two or the last two) in any minor of a matrix \( A \) are consecutive in the original matrix \( A \), it follows that \( \mathbf{P}_1 \) is a hereditary property. (It should be noted that if we delete any arbitrary row and any arbitrary column from a matrix \( A \) satisfying property \( \mathbf{P}_1 \), the resultant matrix may not satisfy property \( \mathbf{P}_1 \).) If the result is not true, then there exists some \( n < \infty \), such that \( \Delta(\mathbf{P}_1) = n \). Let \( C \) be an \( n \times n \) matrix satisfying the property \( \mathbf{P}_1 \), for which none of the optimal tours is pyramidal. Let \( \tau \) be an optimal tour. Let \( \tau^{-1}(n) = a \) and \( \tau(n) = b \). Let \( a < b \) (Else, replace \( \tau \) by \( \tau^{-1} \) and \( C \) by \( C^T \) and continue.). By Lemma 1, every optimal tour corresponding to \( C \) has \( (n-1) \) as one of its peaks. Hence, \( b < n-1 \). Let \( \tau' \) be a tour on \( \{1, 2, \ldots, n\} \) obtained from \( \tau(b) \) by replacing arc \((a, b)\) by path \((a - (n - (n-1)) \cdots b)\). Let us recollect that for any \( 2 < i \leq n \), \( C_{i, i} = R(u, v, i, i) \) for distinct \( u, v < i \), where \( (u - (i - 1) - v) \) is a subpath of \( \tau(i+1) \). It should be noted that if \( b = n - 2 \), then \( C_{i, (n-1)} = R(u, v, n - 1, n - 1) \) for some \( v < (n - 2) \). Thus,

\[
C(\tau) - C(\tau') = \left\{ C^{\tau(b)}(\tau(b)) + \sum_{j=b+1}^{n} C^{\tau(j)} \right\} - \left\{ C^{\tau(b)}(\tau(b)) + \sum_{j=b+1}^{n} R(a, j-1, j, j) \right\}
\]

\[
= \sum_{j=b+1}^{n} C^{\tau(j)} - \sum_{j=b+1}^{n} R(a, j-1, j, j)
\]

\[
= \sum_{j=b+1}^{n-1} R(j, j-1, n, j) + \sum_{j=b+1}^{n-1} C^{\tau(j)}
\]

\[
\geq 0 \quad \text{(From condition (i) of property } \mathbf{P}_1)\]
So, \( \tau' \) is an optimal tour corresponding to \( C \) and does not have \( (n - 1) \) as its peak. We thus have a contradiction. This completes the proof. \( \square \)

It follows from the next result that the Demidenko condition is a special case of property \( P_1 \). (For the statement of the Demidenko condition, please refer to Appendix A.)

**Lemma 2** (Baki [1], Baki and Kabadi [2]). An \( n \times n \) cost matrix, \( C \), satisfies Demidenko condition iff \( C \) and \( C^T \) satisfy the following condition:

\[
\text{for all } 1 \leq i, j < k < u \leq n, i \neq j, R(i, j, k, k) + R(k, k - 1, u, k) \geq 0.
\]

**Proof.** If in the above expression, we substitute \( j = k - 1 \), we get inequality (ii) of the Demidenko conditions. Again, if we substitute in the above inequality, \( i = k - 1 \), we get inequality (i) of the Demidenko condition for \( C^T \).

Conversely, suppose a matrix, \( C \), satisfies the Demidenko condition. Consider any \( \{i, j, k, u\} \) such that \( 1 \leq j < i < k < u \leq n \). Then,

\[
R(i, j, k, k) + R(k, k - 1, u, k) = L(j, k - 1, u) + R(i, j, i + 1, k) + R(i + 1, j, i + 2, k) + \cdots + R(k - 2, j, k - 1, k) \geq 0.
\]

The case \( 1 \leq i < j < k < u \leq n \) can be shown similarly. \( \square \)

We thus have the following corollary:

**Corollary 1.** If cost matrix, \( C \), satisfies the Demidenko condition, then there exists an optimal tour that is pyramidal.

Until 1995, it was a well-known open problem to find a simple and/or short proof of sufficiency of the Demidenko condition [1, 5]. The first such simple and short proof was given in [1] (see also [2]). The above proof of Theorem 1 is a further simplification of the proof in [1, 2] and provides us with the simplest known proof of sufficiency of the Demidenko condition.

We give below a \( 6 \times 6 \) cost matrix which satisfies property \( P_1 \) but violates the Demidenko condition.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]

**Theorem 2.** There is an \( O(n^3) \) scheme to test if a given matrix satisfies property \( P_1 \).
Proof. For any \( 2 < k < n \), let
\[
R^*(k) = \min \{ R(u_k, v_k, k, k) : u_k, v_k < k \}
\]
and for any \( 2 < i < j < n \), let
\[
R^*(i, j) = \min \{ R(j, i, x, j) : j \leq x \leq n \}.
\]
All the values \( \{ R^*(k) : 2 < k < n \} \) and \( \{ R^*(i, j) : 2 < i < j < n \} \) can be calculated in \( O(n^3) \). Testing if a given matrix, \( A \), satisfies property \( P_1 \) is now equivalent to checking if

(i) For each \( 2 < i < j < n \), and distinct \( u_k, v_k < k \) for each \( k \in \{ i, i + 1, \ldots, j \} \),
\[
\sum_{k=1}^{j} R(k, k - 1, x, k) + \sum_{k=1}^{j} R^*(k) + R^*(i, j) \geq 0
\]
and

(ii) For any \( 2 < i < x \leq n \),
\[
R(i, i - 1, x, i) + R^*(i, i) \geq 0.
\]
The complexity of this is \( O(n^3) \). \( \square \)

It is however an open problem to produce a polynomial scheme for testing if, for a given instance of \( TSP \), there exists a numbering of the nodes such that the cost matrix satisfies property \( P_1 \). In fact, this is also true for the special case of the Demidenko condition.

4. A further generalization of Demidenko condition

We shall now use some elementary observations to further generalize the sufficiency condition of the previous section.

For any tour \( \tau \) and any \( i \leq n \), it follows from the definition of \( C^{\tau, i} \), given in Section 2 (see Definition 5), that \( C^{\tau, i} = R(u, v, i, i) \) for distinct \( u, v < i \), where, \( (u - (i + 1) - v) \) is a subpath of \( \tau^{(i+1)} \). Consider any tour \( \tau \) with \( \tau^{-1}(a) = a \) and \( \tau(n) = b \) for some \( a < b < n - 1 \), Then it can be readily seen that there exist \( b = i_0 < i_1 < \cdots < i_r < i_{r+1} (= n) \) for some \( r \geq 0 \) such that for any \( 0 \leq j \leq r \),
either (i) there exists \( u_j < i_j - 1, u_j \neq a \), such that
\[
C^{\tau, k} = R(u_p, k - 1, k, k) \text{ for all } i_j < k < i_{j+1} \text{ and}
\]
\[
C^{\tau, i_j} = R(u_p, v_j, i_p, i_j) \text{ for some } v_j < i_j - 1, v_j \neq b, v_j \neq u_j;
\]
or, (ii) \( C^{\tau, k} = R(i_j - 2, k - 1, k, k) \) for all \( i_j \leq k < i_{j+1} \);
or, (iii) there exists \( v_j < i_j - 1, \ v_j \neq b, \) such that

\[
C^{r,k} = R(k - 1, v_j, k, k)
\]

for all \( i_j < k < i_{j+1} \) and

\[
C^{r,i_j} = R(u_j, v_j, i_j, i_j)
\]

for some \( u_j < i_j - 1, \ u_j \neq a, \ u_j \neq v_j. \)

or, (iv) \( C^{r,k} = R(k - 1, i_j - 2, k, k) \) for all \( i_j \leq k < i_{j+1}. \)

Furthermore, if \( r = 0 \) and \( v_0 = a, \) then \( a = 1. \) (Else, \( \tau^{b-1} \)) will not be a tour on node set \( \{1, \ldots, (b-1)\}. \)

For any \( n \times n \) matrix \( A, \) any \( 2 < i < j < n, \) and any distinct \( u, v < i - 1, \) let us define

\[
S1(u, v, i, j) = R(u, v, i, i) + \sum_{k=i+1}^{j} R(u, k - 1, k, k),
\]

\[
S2(i, j) = \sum_{k=i}^{j} R(i - 2, k - 1, k, k),
\]

\[
S3(u, v, i, j) = R(u, v, i, i) + \sum_{k=i+1}^{j} R(k - 1, v, k, k),
\]

\[
S4(i, j) = \sum_{k=i}^{j} R(k - 1, i - 2, k, k).
\]

For any \( a < b < x < n, \) define a network \( \{G_{a,b,x}, r_{a,b,x}\} \) on node set \( N_{b,x} = \{b, b + 1, \ldots, x\} \) with arc set \( E_{a,b,x} = \{(i, j) : b < i < j < x\} \) and arc weight function \( r_{a,b,x} \) defined as

\[
r_{a,b,x}(b, j) = \min\{\{S1(u, v, b + 1, j), \ S3(u, v, b + 1, j) : u, v \leq b; u \neq v; u \neq a; v \neq b\}, \ S2(b + 1, j),
\]

\[
S4(b + 1, j)\} \forall b < j < x;
\]

\[
r_{a,b,x}(b, x) = \min\{\{S1(u, v, b + 1, x), \ S3(u, v, b + 1, x) : u, v \leq b; u \neq v; u \neq a; v \neq b, \text{ and if } a \neq 1, \text{ then } v \neq a\}, \ S2(b + 1, x), \ S4(b + 1, x)\};
\]

\[
r_{a,b,x}(i, j) = \min\{\{S1(u, v, i + 1, j), \ S3(u, v, i + 1, j) : u, v < i - 1; u \neq v; u \neq a; v \neq b\}, \ S2(i + 1, j), \ S4(i + 1, j)\} \forall b < i < j < x.
\]

Let \( d_{a,b}(x) \) be the length of the shortest path from node \( b \) to node \( x \) in \( \{G_{a,b,x}, r_{a,b,x}\}. \)

We are now ready to define our more general sufficiency condition.

**Definition 7.** An \( n \times n \) cost matrix \( A \) satisfies property \( P_2 \) iff both \( A \) and \( A^T \) satisfy the following:

For any \( a < b < i < n, \)

either (i) \( d_{a,b}(i) + \sum_{k=b+1}^{i} R(k, k - 1, x, k) \geq 0, \) \( \forall i < x \leq n \)

or (ii) \( d_{a,b}(i) + \sum_{k=b+1}^{i-1} R(a, k, k + 1) + R(a, i, i, x) \geq 0, \) \( \forall i < x \leq n. \)
From the complexity of the shortest path problem (see [6]) and using the same arguments as in the proof of Theorem 2, it follows that the property $P_2$ can be verified in $O(n^4 \log n)$.

**Theorem 3.** If cost matrix $C$ satisfies property $P_2$ then there exists an optimal tour that is pyramidal.

*Proof.* In the definition of property $P_2$, for any $a < b < i < n$, the expressions in (i) and (ii) depend only on information corresponding to the rows in the matrix between row $a$ and row $i$. For any minor, $B$, of matrix $A$ containing rows $a$, $b$, and $i$, the set of rows in $B$ between row $a$ and row $i$, is the same as the corresponding set in matrix $A$. Hence, if the matrix $A$ satisfies property $P_2$, then its minors satisfy property $P_2$. Thus, the property $P_2$ is a hereditary property.

If the result is not true, then $\Delta(P_2) = n < \infty$. Let $C$ be an $n \times n$ matrix satisfying the property $P_2$ for which none of the optimal tours is pyramidal. Let $\tau$ be an optimal tour. Let $\tau^{-1}(n) = a$ and $\tau(n) = b$. Let $a < b$ (Else, replace $\tau$ by $\tau^{-1}$ and $C$ by $C^T$ and continue.). By Lemma 1, every optimal tour corresponding to $C$ has $(n - 1)$ as one of its peaks. Hence, $b < n - 1$.

*Case (i):* The triplet $(a, b, n - 1)$ satisfies condition (i) of property $P_2$.

Let $\tau'$ be a tour on $\{1, 2, \ldots, n\}$ obtained from $\tau^{(b)}$ by replacing arc $(a, b)$ by path $(a - n - (n - 1) - \cdots - b)$.

Then,

$$
C(\tau) - C(\tau') = \sum_{j = b + 1}^{n} C^{(b), j} - \sum_{j = b + 1}^{n} R(a, j - 1, j, j)
$$

$$
= \sum_{k = b + 1}^{n - 1} C^{(b), k} + \sum_{k = b + 1}^{n - 1} R(k, k - 1, n, k)
$$

$$
\geq d_{a,b}(n - 1) + \sum_{k = b + 1}^{n - 1} R(k, k - 1, n, k)
$$

$$
\geq 0.
$$

So, $\tau'$ is an optimal tour corresponding to $C$ and does not have $(n - 1)$ as its peak. We thus have a contradiction.

*Case (ii):* The triplet $(a, b, n - 1)$ satisfies condition (ii) of property $P_2$.

Let $\tau'$ be a tour on $\{1, 2, \ldots, n\}$ obtained from $\tau^{(b)}$ by replacing arc $(a, b)$ by path $(a - (b + 1) - (b + 2) - \cdots - n - b)$.

Then,

$$
C(\tau) - C(\tau') = \sum_{k = b + 1}^{n - 1} C^{(b), k} + \sum_{k = b + 1}^{n - 1} R(a, k, k, k + 1)
$$

$$
\geq d_{a,b}(n - 1) + \sum_{k = b + 1}^{n - 1} R(a, k, k, k + 1)
$$

$$
\geq 0.
$$
Hence, \( \tau' \) is an optimal tour corresponding to \( C \) and does not have \((n - 1)\) as its peak. We thus have a contradiction.

This completes the proof. \( \Box \)

The corollary below follows from Theorem 3 and Observation 1.

**Corollary 2.** If the cost matrix \( C \) is such that \( C_k \) satisfies property \( P_2 \) then there exists an optimal tour that is pyramidal.

5. A generalization of Warren's conditions

We shall now present new sufficiency conditions which properly generalize the eight sets of conditions given by Warren [8].

**Definition 8.** An \( n \times n \) cost matrix \( C \) satisfies property \( P_4 \) iff

(i) For any 2 distinct numbers \( i_1, i_2 < n - 1 \)
    and 2 distinct numbers \( j_1, j_2 < n - 1 \)
    such that \( i_u \neq j_v \) for \( u \neq v, u, v \in \{1, 2\} \) and \(|\{i_1, i_2, j_1, j_2\}| \geq 3\)
    let \( C' \) be a \( 4 \times 4 \) submatrix of \( C \) with rows indexed by \( i_1, i_2, n - 1, n \) (in that order) and columns indexed by \( j_1, j_2, n - 1, n \) (in that order).
    Then, the TSP on node set \{1, 2, 3, 4\} with cost matrix \( C' \) has an optimal tour that is pyramidal.

(ii) Every upper minor of \( C \) satisfies condition (i) above.

**Theorem 4.** If cost matrix \( C \), satisfies property \( P_4 \), then there exists an optimal tour that is pyramidal.

*Proof.* If the result is not true, then there exists some \( n < \infty \), such that \( \Delta(P_4) = n \). Let \( C \) be a \( n \times n \) cost matrix satisfying property \( P_4 \) for which there does not exist an optimal tour that is pyramidal. Property \( P_4 \) is obviously upper hereditary. Hence, by Lemma 1, every optimal tour corresponding to \( C \) has \((n - 1)\) as one of its peaks. Consider an optimal tour, \( \tau \). Then \( \tau \) is of the form

\[(n - 1)[a_1, b_1] - n - [a_2, b_2] - n - 1).\]

Remove arcs \{\((n - 1), a_1\), \((b_1, n), (n, a_2), (b_2, (n - 1))\}\) from \( \tau \) to get 2 disjoint paths \([a_1, b_1]\) and \([a_2, b_2]\) in addition to isolated nodes \( n \) and \((n - 1)\). Replace the paths \([a_1, b_1]\) and \([a_2, b_2]\) in \( \tau \) by new nodes \( 1' \) and \( 2' \) to get a tour \( \tau' \) on node set \{1', 2', (n - 1), n\}. Consider the submatrix \( C' \) of \( C \) with rows indexed by \{\(b_1, b_2, (n - 1), n\}\) and columns indexed by \{\(a_1, a_2, (n - 1), n\}\), (in that order). Then, by property \( P_4 \), there exists an optimal tour \( \tau'' \) on \{1', 2', (n - 1), n\} corresponding to cost matrix \( C' \), such that \((n - 1)\) is adjacent to \( n \). In \( \tau'' \), replace nodes \( 1' \) and \( 2' \) by paths \([a_1, b_1]\) and \([a_2, b_2]\) to get a tour \( \tau''' \) on node set \{1, 2, \ldots, n\}. Then,

\[C'(\tau''') - C(\tau) = C'(\tau') - C(\tau') \leq 0.\]
Thus, $\tau''$ is an optimal tour corresponding to $C$ and does not have $(n - 1)$ as its peak. We thus, have a contradiction. This proves the theorem. □

**Definition 9.** An $n \times n$ matrix $C$ satisfies property $P_4$ if its reverse permutation, $C_R$, satisfies property $P_4$.

The Corollary below follows from Theorem 4 and Observation 1.

**Corollary 3.** If cost matrix $C$ satisfies property $P_4$, then there exists an optimal tour corresponding to $C$ that is pyramidal.

As indicated before, the most general sufficient conditions for existence of a pyramidal optimal tour known so far [1] are (i) the Demidenko condition [3], (ii) the four sets of conditions by Baki and Kabadi [1, 2] and (iii) the eight sets of conditions by Warren [8]. (All these conditions are given in the Appendix A.)

It is easy to verify that Warren's Conditions I, III, V, and VII are special cases of the property $P_4$ and Warren's conditions II, IV, VI and VIII, are special cases of the property $P_4$.

Furthermore, the complexity of testing property $P_4$ can be easily shown to be $O(n^3)$, which is the same as that of each of the eight conditions of Warren.

We shall now give a matrix satisfying property $P_4$ but none of the other previously known conditions. Thus, consider the following matrix, $C'$:

$$
\begin{array}{cccccc}
0 & 1 & -1 & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 \\
\end{array}
$$

It may be observed that this matrix satisfies property $P_4$. The fact that the matrix does not satisfy condition sets I, III, V and VII of Warren follows respectively from (i), (ii), (iii) and (iv) below:-

(i) $C(1, 2) + C(2, 3) + C(3, 4) = 0 > -1 = C(1, 3) + C(3, 2) + C(2, 4),$

$C(1, 2) + C(5, 3) + C(3, 1) = 1 > 0 = C(1, 3) + C(3, 2) + C(5, 1);$

(ii) $C(1, 2) + C(2, 3) + C(3, 4) = 0 > -1 = C(1, 3) + C(3, 2) + C(2, 4);$

(iii) $C(1, 2) + C(4, 1) + C(3, 4) = 2 > -2 = C(1, 4) + C(3, 1) + C(4, 2);$

(iv) $C(3, 2) + C(5, 1) + C(4, 5) = 1 > -1 = C(3, 5) + C(4, 1) + C(5, 2),$

$C(3, 2) + C(5, 4) + C(1, 5) = 0 > -2 = C(3, 5) + C(1, 4) + C(5, 2).$

Since, $C(1, 3) + C(5, 2) - C(1, 2) - C(5, 3) = -4 < 0$, the matrix $C$ does not satisfy Demidenko condition, and condition sets I and III of Baki–Kabadi. Since $C(1, 4) + C(3, 2) - C(1, 2) - C(3, 4) = -1 < 0$, the matrix does not satisfy condition sets II and IV of Baki–Kabadi. Using $\{i_1, i_2\}$
\(= \{2, 3\}\) and \(\{j_1, j_2\} = \{2, 1\}\), it may be verified that we get a violation of case (i) of definition of property \(P_4\). Thus, the matrix does not satisfy property \(P_4\) (and hence, it does not satisfy Warren conditions II, IV, V and VIII).

If \(C_R\) is the reverse permutation of \(C\), then \(C_R\) satisfies property \(P_4\) and it can be shown similar to above that \(C_R\) satisfies none of the other previously known sufficiency conditions.

The property \(P_4\) can be generalized to the following property \(P_k\) for any \(k \geq 4\).

**Definition 10.** For an integer \(k \geq 4\), an \(n \times n\) cost matrix \(C\) satisfies property \(P_k\) iff

(i) For any \(k - 2\) distinct numbers \(i_1, i_2, \ldots, i_{k-2} < n - 1\) and \(k - 2\) distinct numbers \(j_1, j_2, \ldots, j_{k-2} < n - 1\) such that \(i_u \neq j_v\) for all \(u \neq v\) and \(|\{i_1, i_2, \ldots, i_{k-2}, j_1, j_2, \ldots, j_{k-2}\}| \geq k - 1\)

let \(C'\) be a \(k \times k\) submatrix of \(C\) with rows indexed by \(i_1, i_2, \ldots, i_{k-2}, (n - 1), n\) (in that order) and columns indexed by \(j_1, j_2, \ldots, j_{k-2}, (n - 1), n\) (in that order). Then, the TSP on node set \(\{1, 2, \ldots, k\}\) with cost matrix \(C'\) has an optimal tour with node \((k - 1)\) adjacent to node \(k\).

(ii) For any \(r < k - 1\) and \(i, j > r, i \neq j\), TSP on node set \(\{1, 2, \ldots, r + 1\}\) with cost matrix \(C^{i,i,j,r}\) has an optimal tour with node \(r\) adjacent to node \((r + 1)\).

(iii) Every upper minor of \(C\) satisfies conditions (i) and (ii) above.

For any fixed \(k\), property \(P_k\) can be tested in polynomial time. However, the complexity of the testing scheme is exponential in \(k\). Hence, for large values of \(k\), the testing scheme will not be practical and interest in the corresponding property \(P_k\) is purely of theoretical nature.

**Theorem 5.** If cost matrix \(C\), satisfies property \(P_k\) for some integer \(k \geq 4\), then there exists an optimal tour that is pyramidal.

**Proof.** If the result is not true, then there exists a \(k \geq 4\) and some \(n < \infty\), such that \(\Delta(P_k) = n\).

It follows from the definition of the property \(P_k\), that \(n > k\). Let \(C\) be a \(n \times n\) cost matrix satisfying property \(P_k\) and for which there does not exist an optimal tour that is pyramidal. Since property \(P_k\) is hereditary, by Lemma 1, every optimal tour corresponding to \(C\) has \((n - 1)\) as one of its peaks.

Consider an optimal tour, \(\tau\). Then \(\tau\) is of the form:-

\[((n - 1) - [a_1, b_1] - n - [a_2, b_2] - (n - 1)).\]

Remove arcs \(\{(n - 1), a_1\}, (b_1, n), (n, a_2), (b_2, (n - 1)\}\) and an additional \((k - 4)\) arcs from \(\tau\) to get \((k - 2)\) disjoint paths \(\mu_1, \mu_2, \ldots, \mu_{k-2}\) in addition to isolated nodes \(n\) and \((n - 1)\). For \(i \in \{1, \ldots, k - 2\}\) let \(\mu_i = [u_i, v_i]\) for some \(u_i, v_i \leq n - 2\). Replace paths \(\mu_1, \mu_2, \ldots, \mu_{k-2}\) in \(\tau\) by new nodes \(1', 2', \ldots, (k - 2)'\) to get a tour \(\tau'\) on node set \(\{1', 2', \ldots, (k - 2)', (n - 1), n\}\).

Consider the submatrix \(C'\) of \(C\) with rows indexed by \(\{v_1, v_2, \ldots, v_{k-2}, (n - 1), n\}\) and columns indexed by \(\{u_1, u_2, \ldots, u_{k-2}, (n - 1), n\}\) (in that order). Then, by property \(P_k\), there exists an optimal tour \(\tau''\) on \(\{1', 2', \ldots, (k - 2)r, (n - 1), n\}\) corresponding to cost matrix \(C'\), such that \((n - 1)\) is adjacent to \(n\). In \(\tau''\), replace nodes \(1', 2', \ldots, (k - 2)'\) by paths \(\mu_1, \mu_2, \ldots, \mu_{k-2}\) to get a tour \(\tau'''\) on
node set \{1, 2, \ldots, n\}. Then,
\[ C'(\tau'') - C(\tau) = C'(\tau'') - C(\tau') \leq 0. \]

Thus, \(\tau''\) is an optimal tour corresponding to \(C\) and does not have \((n - 1)\) as its peak. We thus, have a contradiction. This proves the theorem. ☐

**Definition 11.** For any integer \(k \geq 4\), an \(n \times n\) matrix \(C\) satisfies property \(P_k\) if its reverse permutation, \(C_R\), satisfies property \(P_k\).

The corollary below follows from Theorem 5 and Observation 1.

**Corollary 4.** If cost matrix \(C\) satisfies property \(P_k\), then there exists an optimal tour corresponding to \(C\) that is pyramidal.

**Theorem 6.** For any \(k \geq 4\), if a matrix satisfies property \(P_k\), then it satisfies property \(P_{k+1}\).

**Proof.** Suppose for some \(k \geq 4\), there exists a \(n \times n\) matrix \(C\), that satisfies property \(P_k\) but does not satisfy property \(P_{k+1}\). Then, there exists a \((k + 1) \times (k + 1)\) submatrix, \(C'\) of \(C\) of the type in case (i) of the definition of \(P_{k+1}\) such that no optimal tour of \(\{1, 2, \ldots, (k + 1)\}\) corresponding to \(C'\) has node \(k\) adjacent to node \((k + 1)\).

Consider one such optimal tour \(\tau\). Then \(\tau\) has a subpath of the type \(((k + 1) - \mu - k)\) or \((k - \mu - (k + 1))\) such that \(\mu\) is a path of length at least 2. Let \((u, v)\) be an arc in \(\mu\) for some \(u, v \in \{1, 2, \ldots, k - 1\}\). Let \(x = \min\{u, v\}\) and \(y = \max\{u, v\}\). In \(\tau\), replace the arc \((u, v)\) by node \(x\) and for each \(i > y\), renumber node \(i\) as \((i - 1)\) to obtain a tour \(\tau'\) on \(\{1, 2, \ldots, k\}\). Define a \(k \times k\) matrix \(C''\) as follows:

for \(i < y\), \(i \neq x\) \[ C''_{i+1} = C'_{i+1} \]

and \(C''_{i} = C'_{i}\) and \(C''_{i+1} = C'_{i+1}\)

Then \(C''\) is of the type in case (i) of definition of \(P_k\) and so there exists an optimal tour, \(\tau''\), on \(\{1, 2, \ldots, k\}\) corresponding to cost matrix \(C''\) such that node \((k - 1)\) is adjacent to node \(k\). In \(\tau''\), for each \(i \geq y\), renumber node \(i\) as \((i + 1)\), and replace node \(x\) by arc \((u, v)\) to get a tour \(\tau'''\) on node set \(\{1, 2, \ldots, (k + 1)\}\). Then,
\[ C'(\tau''') - C'(\tau) = C''(\tau') - C''(\tau') \leq 0. \]

So, \(\tau''''\) is an optimal tour corresponding to cost matrix \(C\) with node \(k\) adjacent to node \((k + 1)\). We thus have a contradiction. This proves the theorem. ☐

It is easy to produce examples of matrices that satisfy property \(P_{k+1}\) but not property \(P_k\) for any \(k \geq 4\). For \(k > 8\), the properties \(P_k\) and \(P_{k+1}\) can be generalized to a single hereditary property that is sufficient for existence of a pyramidal tour. However, since the complexity of scheme for testing
property $P_k$ this exponential in $k$, for large values of $k$, property $P_k$ cannot be tested efficiently in practice (we believe that the scheme will be impractical for $k > 4$). Interest in property $P_k$ for $k > 4$ is, therefore, purely of theoretical nature.

6. A common generalization

We shall now combine the results of Sections 3–5 to obtain a common, general, polynomially testable sufficiency condition.

**Definition 12.** An $n \times n$ cost matrix $A$ satisfies property $P$ iff

either

$\forall a < b < (n - 1)$, each of $A$ and $A^T$ satisfies at least one of the conditions $Q_{a,b}^1$ and $Q_{a,b}^2$ below and

every upper minor of $A$ satisfies property $P$

or

$\forall a < b < (n - 1)$, each of $A_R$ and $A_R^T$ satisfies at least one of the conditions $Q_{a,b}^1$ and $Q_{a,b}^2$

and

every lower minor of $A$ satisfies property $P$. For some $a < b < (n - 1)$, an $n \times n$ matrix, $A$ satisfies property $Q_{a,b}^1$ iff

$$d_{a,b}(n - 1) + \sum_{k = b + 1}^{n - 1} R(k, k - 1, n, k) \geq 0$$

or

$$d_{a,b}(n - 1) + \sum_{k = b + 1}^{n - 1} R(a, k, k + 1) \geq 0,$$

where the terms $d_{a,b}(n - 1)$ and $R(i, j, u, v)$ are as defined before.

For some $a < b < (n - 1)$, an $n \times n$ matrix $A$ satisfies property $Q_{a,b}^2$ iff

For any distinct numbers $i_2, j_1 < n - 1$, such that $i_2 \neq a$ and $j_1 \neq b$, let $C'$ be a $4 \times 4$ submatrix of $C$ with rows indexed by $a, i_2, (n - 1), n$ (in that order) and columns indexed by $j_1, b, (n - 1), n$ (in that order). Then, the TSP on node set $\{1, 2, 3, 4\}$ with cost matrix $C'$ has an optimal tour that is pyramidal.

**Theorem 7.** If the cost matrix $C$ satisfies property $P$, then there exists an optimal tour that is pyramidal.

**Proof.** It follows from the fact that $P_2$ and $P_4$ are hereditary properties that the property $P$ is also a hereditary property. If the result is not true, then there exists some $n < \infty$, such that $\Delta(P_n) = n$. Let $C$ be an $n \times n$ cost matrix satisfying property $P$ for which none of the optimal tours is pyramidal. Let $\tau$ be an optimal tour. Suppose for each $a < b < (n - 1)$, each of $C$ and $C^T$ satisfies at
least one of the conditions \( Q_{a,b}^1 \) and \( Q_{a,b}^2 \). (Else, replace \( C \) by \( C_R \) and \( \tau \) by \( \tau_R \). Let \( \tau^{-1}(n) = u \)
and \( \tau(n) = v \). Let \( u < v \). (Else, replace \( \tau \) by \( \tau^{-1} \) and \( C \) by \( C^T \) and continue.) By Lemma 1, every
optimal tour corresponding to \( C \) has \((n - 1) \) as one of its peaks. Hence, \( v < n - 1 \). If \( C \) satisfies the
condition \( Q_{a,v}^1 \) then the result follows as in the proof of Theorem 3. If \( C \) satisfies the condition
\( Q_{u,v}^2 \) then the result follows as in the proof of Theorem 4. □

It is easy to see that complexity of testing property \( P \) is \( O(n^7) \), which is the same as the complexity of
testing Warren’s conditions. The bottleneck in testing for property \( P \) is the test for property \( Q_{a,b}^2 \).
The following \( 6 \times 6 \) matrix has been given by Warren [8] as an example of a pyramidal
solvably case that does not satisfy any of the previously known sufficiency conditions. It is easy to
verify that it satisfies property \( P \).

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Appendix A

Demidenko condition [3]. For any \( 1 \leq i < j < l - 1 < n \),

Let \( L(i, j, l) = C(j, j + 1) + C(l, j) + C(j + 1, i) - C(j, i) - C(l, j + 1) - C(j + 1, j) \).

Then, both \( C \) and \( C^T \) satisfy the following:

(i) for any \( i < j < k < l \), \( R(i, j, k, k) \geq 0 \),

(ii) for any \( i < j < l - 1 < n \), \( L(i, j, l) \geq 0 \).

Baki–Kabadi Conditions [1, 2].

Set I: \( \forall \ 1 \leq i < j < j + 1 < u, v \),

(i) \( C(i, u) + C(j, j + 1) + C(v, j) - C(j, u) - C(v, j + 1) - C(i, j) \geq 0 \),

(ii) \( C(u, i) + C(j, v) + C(j + 1, j) - C(j, i) - C(u, j) - C(j + 1, v) \geq 0 \),

(iii) \( C(i, j + 1) + C(u, j) - C(i, j) - C(u, j + 1) \geq 0 \),

(iv) \( C(j, u) + C(j + 1, i) - C(j, i) - C(j + 1, u) \geq 0 \).

Set II: \( \forall \ 1 \leq i < j < j + 1 < u, v \)

(i) \( C(i, u) + C(j, j + 1) + C(v, j) - C(j, u) - C(v, j + 1) - C(i, j) \geq 0 \),

(ii) \( C(u, i) + C(j, v) + C(j + 1, j) - C(j, i) - C(u, j) - C(j + 1, v) \geq 0 \),
(iii) \( C(i, u) + C(j + 1, j) - C(i, j) - C(j + 1, u) \geq 0 \),

(iv) \( C(j, j + 1) + C(u, i) - C(j, i) - C(u, j + 1) \geq 0 \).

Set III: \( \forall 1 \leq i, j < u < v - 1 \leq n - 1 \)

(i) \( C(i, v) + C(u, u + 1) + C(u + 1, j) - C(i, u + 1) - C(u + 1, v) - C(u, j) \geq 0 \),

(ii) \( C(v, i) + C(j, u + 1) + C(u + 1, u) - C(u + 1, i) - C(v, u + 1) - C(j, u) \geq 0 \),

(iii) \( C(j, u + 1) + C(v, u) - C(j, u) - C(v, u + 1) \geq 0 \),

(iv) \( C(u, v) + C(u + 1, j) - C(u, j) - C(u + 1, v) \geq 0 \).

Set IV: \( \forall 1 \leq i, j < u < v - 1 \leq n - 1 \)

(i) \( C(i, v) + C(u, u + 1) + C(u + 1, j) - C(i, u + 1) - C(u + 1, v) - C(u, j) \geq 0 \),

(ii) \( C(v, i) + C(j, u + 1) + C(u + 1, u) - C(u + 1, i) - C(v, u + 1) - C(j, u) \geq 0 \),

(iii) \( C(j, v) + C(u + 1, u) - C(j, u) - C(u + 1, v) \geq 0 \),

(iv) \( C(u, u + 1) + C(v, j) - C(u, j) - C(v, u + 1) \geq 0 \).

Warren Conditions [8].

Set I: \( \forall 1 \leq i, j, q < p < k \leq n, i \neq j \) and \( j \neq q \),

\[
1 \leq i, j, k < p < q \leq n, i \neq j \quad \text{and} \quad i \neq k,
\]

\[
C(i, j) + C(k, p) + C(p, q) \leq C(i, p) + C(p, j) + C(k, q).
\]

Set II: \( \forall 1 \leq k < p < i, j, q \leq n, i \neq j \) and \( j \neq q \),

or \( 1 \leq q < p \leq i, j, k \leq n, i \neq j \) and \( i \neq k \),

\[
C(i, j) + C(k, p) + C(p, q) \leq C(i, p) + C(p, j) + C(k, q).
\]

Set III: \( \forall 1 \leq i, j, k < p < q = p + 1 \leq n \) and \( i \neq j \),

\[
C(i, j) + C(k, p) + C(p, q) \leq C(i, p) + C(p, j) + C(k, q) \quad \text{for} \quad i \neq k,
\]

\[
C(i, j) + C(q, p) + C(p, k) \leq C(i, p) + C(p, j) + C(q, k) \quad \text{for} \quad j \neq k.
\]

Set IV: \( \forall 1 \leq q = p - 1 \leq p \leq i, j, k \leq n \) and \( i \neq j \),

\[
C(i, j) + C(k, p) + C(p, q) \leq C(i, p) + C(p, j) + C(k, q) \quad \text{for} \quad i \neq k,
\]

\[
C(i, j) + C(q, p) + C(p, k) \leq C(i, p) + C(p, j) + C(q, k) \quad \text{for} \quad j \neq k.
\]

Set V: \( \forall 1 \leq i, j, k < p < p + 1 \leq q \leq n, i \neq k \) and \( j \neq k \),

\[
C(i, k) + C(q, j) + C(p, p + 1) \leq C(i, p + 1) + C(p, j) + C(q, k),
\]

\[
C(k, i) + C(j, q) + C(p + 1, p) \leq C(p + 1, i) + C(j, p) + C(k, q).
\]
Set VI: \( \forall 1 \leq q \leq p - 1 < p < i, j, k \leq n, i \neq k \) and \( j \neq k \),
\[
C(i, k) + C(q, j) + C(p, p - 1) \leq C(i, p - 1) + C(p, j) + C(q, k),
\]
\[
C(k, i) + C(j, q) + C(p - 1, p) \leq C(p - 1, i) + C(j, p) + C(k, q).
\]
Set VII: \( \forall 1 \leq i, j, k < p < r, s \leq n \) and \( i \neq k \),
\[
C(i, k) + C(s, j) + C(p, r) \leq C(i, r) + C(p, j) + C(s, k) \quad \text{for} \ s \neq k,
\]
or,
\[
C(i, k) + C(s, p) + C(j, r) \leq C(i, r) + C(j, p) + C(s, k) \quad \text{for} \ i \neq j.
\]
Set VIII: \( \forall 1 \leq r, s < p < i, j, k \leq n \) and \( i \neq k \),
\[
C(i, k) + C(s, j) + C(p, r) \leq C(i, r) + C(p, j) + C(s, k) \quad \text{for} \ s \neq k,
\]
or,
\[
C(i, k) + C(s, p) + C(j, r) \leq C(i, r) + C(j, p) + C(s, k) \quad \text{for} \ i \neq j.
\]
Van Der Veen condition [7].

(i) Cost matrix \( C \) is symmetric and
(ii) for any \( 1 \leq i < j < k \leq n \),
\[
C(i, k + 1) + C(j + 1, j) - C(i, j) - C(j + 1, k + 1) \geq 0,
\]
\[
C(j, j + 1) + C(k + 1, i) - C(j, i) - C(k + 1, j + 1) \geq 0.
\]

References