Relativity in Introductory Physics

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Abstract

A century after its formulation by Einstein, it is time to incorporate special relativity early in the physics curriculum. The approach advocated here employs a simple algebraic extension of vector formalism that generates Minkowski spacetime, displays covariant symmetries, and enables calculations of boosts and spatial rotations without matrices or tensors. The approach is part of a comprehensive geometric algebra with applications in many areas of physics, but only an intuitive subset is needed at the introductory level. The approach and some of its extensions are given here and illustrated with insights into the geometry of spacetime.

I. INTRODUCTION

Special relativity, which has now been with us for about one century, presents a new world view or *paradigm*[1] of physics. It revises the concepts of time and space from those assumed in Newtonian and Galilean physics, with consequent changes, for example, in what we mean by simultaneity and how we "add" velocities. It also establishes the framework for fundamental symmetries of electromagnetic phenomena and much of modern physics. Such symmetries provide new approaches to many problems, simplifying many computations, and they are frequently important even at low ("nonrelativistic") velocities, as demonstrated below. Yet, it appears that the physics community has still not completed the paradigm shift: relativity is commonly taught as a complicating correction to Newtonian mechanics, and if elementary electromagnetic theory texts mention it at all, it is usually only later in the text after the basic laws and phenomena have been discussed. Is it not time to integrate relativity more tightly into the early physics curriculum?

One reason for delaying the introduction of relativity in the curriculum can be traced to the conceptual inertia of the educational process: those teaching today learned Newtonian mechanics before relativity and typically feel that a good grounding in Galilean transformations is needed before Lorentz transformations can be understood. It may be argued that the remnants of Aristotelian notions must be cleared away before students are faced with concepts such as the observer-dependence of space and time. However, the relativistic approach is the correct one, and in some respects it may be more intuitive to the beginner than Newtonian mechanics. It is doubtful that it serves the students' best interests to ingrain faulty concepts such as universal time and instantaneous interactions at a distance.

An important practical reason for delaying the introduction of relativity is mathematical. Introductory treatments of relativity that go beyond basic concepts to practical calculations almost all use matrices or tensors at the expense of the vector notation common in Newtonian physics. Matrix and tensor elements are much less effective than vectors at conveying the geometry that is so critical for our physical understanding. (Some efforts to use the geometrical power of differential forms has been made, but they face a conceptual barrier in the abstraction of vector concepts and have been largely reserved to treatments of general relativity.)

This paper advocates an alternative treatment: an algebraic approach based on a simple extension of vectors in physical space. By replacing the dot and cross products of vectors with a simpler but more general associative product, one is led to add scalar time components to vectors. Such objects form a four-dimensional linear space with Minkowski spacetime metric. The algebra that results allows relativistic calculations while avoiding matrices and tensors (although these can be readily derived in the approach). The algebraic approach advocated here for introductory courses is a subset of the powerful and well-developed covariant formalism of Clifford's geometric algebra of physical space (APS).[2–4] APS is the Clifford or geometric algebra of three-dimensional Euclidean space, sometimes denoted $C\ell_3$ or $\mathcal{G}_3$. It is isomorphic (equivalent in structure) to complex quaternions, the algebra of Pauli spin matrices, and the even subalgebra of Hestenes' spacetime algebra (STA).[5, 6]

The following Section emphasizes the importance of relativistic symmetries even at low velocities. In Section III, we introduce the basic algebra needed to compute Lorentz transformations and give several examples appropriate for an introductory course. A discussion of the new geometrical concept of a bivector and its advantages over the more traditional cross product is given in Section IV. Extensions to rotor and spinor representations for use in later courses are discussed in Section V. APS is briefly compared to an alternative algebraic approach using STA in Section VI, and conclusions are summarized in the final Section. While the paper is largely pedagogical, several new results that flow from the APS approach are presented. These include a simple formulation in Subsection III-C of relations between the Minkowskian geometry of spacetime vectors and the lengths and angles actually measured (in a Euclidean sense) on a spacetime diagram, a derivation of reduced algebraic forms for Lorentz rotations in arbitrary spacetime planes Subsection V-B, and the demonstration in Subsection V-C that any boost of an electromagnetic plane wave is equivalent to a spatial rotation and dilation.

II. RELATIVITY AT LOW VELOCITIES

The importance of relativity is well appreciated for understanding Einstein's mass-energy relation or for correctly computing such values as threshold energies for particle production in high-velocity collisions. Less appreciated is the power of relativistic symmetries at low ("nonrelativistic") velocities, particularly in electromagnetic theory. One fruitful example is the understanding that the electric and magnetic fields are aspects of a single covariant electromagnetic field. For example, the electric field lines of a moving charge sweep out spatial planes that represent its magnetic field. The magnetic field is the vector normal (dual) to the planes (see Fig. 1), as discussed more fully in Subsection IV.B. The magnetic field that necessarily accompanies the oscillating electric field in an electromagnetic plane wave has a similar origin. Indeed, the geometric interpretation of the magnetic field as a spatial plane and its connection to the electric field can help to demystify many relations involving **B**.

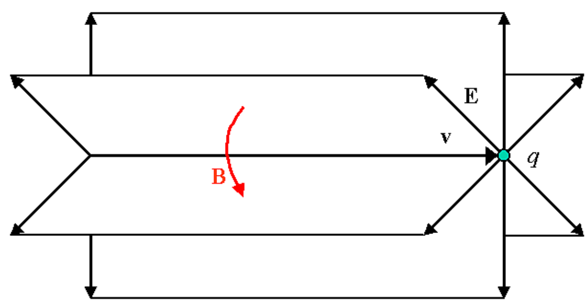

FIG. 1: The electric field lines of a moving charge q sweep out spatial planes that represent the magnetic field **B** .

There are also many cases in which a relativistic transformation can simplify a computation and clarify its significance. The following examples will serve to illustrate that the importance of relativity is not restricted to exotic phenomena that occur only with particle velocities close to the speed of light.

To calculate the effect of a plane wave incident at an oblique angle on the plane surface of a good conductor, we can compute the simpler case of normal incidence and then apply a boost (velocity transformation) in the plane of the conductor (see Fig. 2). This can be extended to the calculation of wave-guide modes by boosting standing waves.[2]

We return to these examples at the end of this paper and show that the boost of a propagating plane wave and of its wave vector in spacetime is equivalent to a rotation and dilation (rescaling).

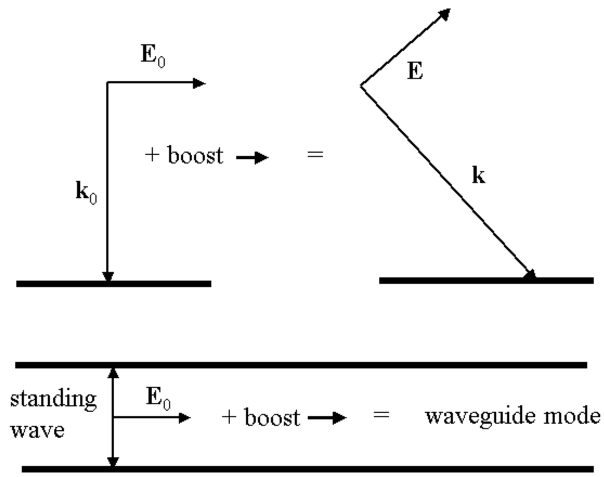

FIG. 2: When propagating plane waves are boosted, they are simply rotated and dilated. As a result, waves obliquely incident on a conductor can be obtained by boosting normally incident ones. Similarly, waveguide modes can be found by boosting standing waves.

III. THE ALGEBRAIC APPROACH

The key step in the algebraic method is to introduce an associative product of vectors. This product, called a *geometric product*, is defined by the axiom that any vector $\mathbf{p}$ times itself is the scalar product $\mathbf{p} \cdot \mathbf{p}$, that is, the square length of $\mathbf{p}$:

$$\mathbf{p}^2 = \mathbf{p}\mathbf{p} = \mathbf{p} \cdot \mathbf{p} \ . \tag{1}$$

The square length of a vector $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ in (flat, Euclidean) physical space is given by the orthonormality of the basis vectors $\mathbf{e}_j$ to be the Pythagorean value $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$. Multiplication of a vector $\mathbf{p}$ by a (real) scalar α scales its length, and the two vectors $\mathbf{p}$ and $\alpha\mathbf{p}$ are said to be *aligned.* It follows that the geometric product of any two aligned vectors is their scalar product. The product of nonaligned vectors will be considered below.

The product of elements in APS behaves like the product of square matrices. There are in fact infinitely many matrix representations of APS, but explicit matrices are never needed. The existence of a single matrix representation proves that APS exists as a consistent algebra, and once its existence has been accepted, only the algebra itself is important.

Those familiar with matrices but uncomfortable with vector algebras may find it initially reassuring to represent the basis vectors $\mathbf{e}_j$ by the three 2×2 Pauli spin matrices σ_j. The vector-space properties of physical space, namely vector addition, scalar products of vectors, and multiplication of vectors by scalars, then work just as well as when the $\mathbf{e}_j$ are thought of as three-dimensional column or row vectors, as is common in elementary vector treatments. However, the alternative representation of vectors as 2×2 matrices also leads naturally to (i) an associative multiplication of vectors, and (ii) the addition of a fourth dimension proportional to the unit matrix $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is linearly independent of the σ_j. A 2×2 representation of the full APS is generated from products of the matrix representations of vectors. Every element is then some 2×2 matrix, and scalars in particular are represented by the scalar value times σ_0.[7] Generally, however, matrices can and probably should be avoided entirely when introducing the algebra at the first- or second-year university level.

The geometric product is the only product needed in the algebra. As we will see below, both the dot and cross product of vectors can be expressed in terms of it. Like the cross product, the geometric product of nonaligned vectors does not commute. This is seen by setting $\mathbf{p} = \mathbf{q} + \mathbf{r}$ in the axiom (1) to get

$$(\mathbf{q} + \mathbf{r})(\mathbf{q} + \mathbf{r}) = \mathbf{q}^2 + \mathbf{rq} + \mathbf{qr} + \mathbf{r}^2 = \mathbf{q} \cdot \mathbf{q} + 2\mathbf{r} \cdot \mathbf{q} + \mathbf{r} \cdot \mathbf{r} \ ,$$

which with the help of the axiom gives an expression for the dot product:

$$\mathbf{r} \cdot \mathbf{q} = \frac{1}{2}(\mathbf{rq} + \mathbf{qr}) \, . \tag{2}$$

If $\mathbf{r}$ and $\mathbf{q}$ are perpendicular, this expression vanishes. Thus, while aligned vectors commute, perpendicular vectors anticommute. Unlike the dot and cross products, the geometric product is associative and invertible: if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are any vectors,

$$(\mathbf{pq})\mathbf{r} = \mathbf{p}(\mathbf{qr}) \equiv \mathbf{pqr}$$

and

$$\mathbf{p}^{-1} = \frac{\mathbf{p}}{\mathbf{p}^2} \ .$$

The additional effort required to teach the geometric product is more than compensated for by avoiding the confusion caused by the non-associative cross product. Furthermore, unlike the cross product, the geometric product is easily extended to spaces of more than three dimensions.

A. Spacetime Vectors

A point r in spacetime depends on the time t as well as on the spatial position $\mathbf{r}$ in physical space. Now elements of APS may be sums of vectors and their products, and as we have seen, those products include real scalars. We might therefore try to represent r in APS by the sum $r = ct + \mathbf{r}$. (The factor of the speed of light c is included to ensure that all components of r have the same dimensions.) In Clifford algebras, the sum of a scalar and a vector is commonly called a *paravector*,[8–12]. We adopt this name here to distinguish it from a vector in three-dimensional physical space. A displacement of r is then a spacetime vector represented by

$$dr = cdt + d\mathbf{r}.$$

Other spacetime vectors are similarly represented, for example the spacetime momentum of a particle is a paravector

$$p = p^0 + \mathbf{p} \ ,$$

where $E = p^0 c$ is the energy and $\mathbf{p}$ the spatial momentum. The sum is analogous to the sum of a real number and an imaginary number to form a complex number.

Whether or not paravectors in APS can represent spacetime vectors depends on how the square length of a paravector is determined. The square length of a vector $\mathbf{p}$ is simply $\mathbf{p}^2$, but the square of a paravector p is generally not a scalar. The analogy to complex numbers suggests that we need to multiply p by a conjugate, and what we need is called the *Clifford conjugate* (or bar conjugate) $\bar{p} = p^0 - \mathbf{p}$, which changes the sign of the vector part. The product

$$p\bar{p} = \left(p^0 + \mathbf{p}\right)\left(p^0 - \mathbf{p}\right) = \left(p^0\right)^2 - \mathbf{p}^2 = \bar{p}p \tag{3}$$

is always a scalar and can be taken as the "square length" of p. The minus sign, which appears naturally here, dictates that paravector space has the geometry of Minkowski spacetime. It marks an important departure from the Euclidean space of $\mathbf{p}$ since the "square length" of p and $\bar{p}$ can be either positive, negative, or zero. By replacing p by $q + r$, the scalar product of two distinct paravectors q, r is found. It is the scalar part of the geometric product $q\bar{r}$:

$$\langle q\bar{r}\rangle_S \equiv \frac{1}{2}\left(q\bar{r} + r\bar{q}\right) = q^0 r^0 - \mathbf{q}\cdot\mathbf{r} = \langle r\bar{q}\rangle_S \ . \tag{4}$$

Note that $\overline{q\bar{r}} = r\bar{q}$. Paravectors q and r are said to be *orthogonal* if $\langle q\bar{r}\rangle_S = 0$. As long as

$p\bar{p} \neq 0$, p has an inverse

$$p^{-1} = \bar{p}/\left(p\bar{p}\right). \tag{5}$$

This is similar to the inverse of a complex number, but there is now a new possibility: if $p\bar{p} = 0$ but $p \neq 0$, then p is *null* and has no inverse. Null elements are orthogonal to themselves. They arise for travel at the speed of light.

The paravector basis is four-dimensional, and we define an orthonormal set $\{e_0, e_1, e_2, e_3\}$ comprising one unit paravector along the time direction $e_0 \equiv 1$ plus the three orthogonal paravectors $e_j \equiv \mathbf{e}_j$ along the spatial axes. Their scalar products give elements of what is known as the *Minkowski spacetime metric tensor*

$$\langle e_\mu \bar{e}_\nu \rangle_S = \begin{cases} 1, & \mu = \nu = 0 \\ -1, & \mu = \nu = 1,2,3 \\ 0, & \mu \neq \nu \end{cases}. \tag{6}$$

B. Lorentz Transformations

An important paradigm of relativity is that space and time are not absolute but are mixed by physical *Lorentz transformations,* which may be viewed as rotations in spacetime. The square length of a spacetime vector is invariant under such rotations. If $dr = cdt + d\mathbf{r}$ is the displacement of a particle, the Lorentz-invariant square length of the displacement is

$$dr\ d\bar{r} = c^2 dt^2 - d\mathbf{r}^2 = c^2 d\tau^2. \tag{7}$$

It suggests the definition of a Lorentz-invariant proper time τ as the time in the commoving inertial frame of the particle, where $d\mathbf{r} = 0$. The dimensionless proper velocity is defined as

$$\begin{aligned} u = \frac{dr}{cd\tau} &= \frac{dt}{d\tau}\left(1 + \frac{d\mathbf{r}}{cdt}\right) \\ &= \gamma\left(1 + \mathbf{v}/c\right), \end{aligned} \tag{8}$$

where $\gamma = dt/d\tau$ is its *time-dilation* factor, and $\mathbf{v} = d\mathbf{r}/dt$ is its coordinate velocity. Since $cd\tau$ is an invariant interval in spacetime, u transforms in the same way as dr and is by definition *unimodular,* that is of unit length:

$$u\bar{u} = 1. \tag{9}$$

It follows immediately that u and $\bar{u}$ are inverses of each other and that

$$\gamma = \frac{dt}{d\tau} = \left[1 - \left(\frac{\mathbf{v}}{c}\right)^2\right]^{-1/2}. \tag{10}$$

A simple Lorentz rotation is a rotation in a single spacetime plane of two dimensions. Any Lorentz rotation can be built up of products of such simple rotations. If the rotation plane comprises two spatial directions, the transformation is a common spatial rotation. If, instead, one of the directions in the rotation plane is the scalar time axis, the transformation is a boost (velocity transformation). A simple rotation mixes the components of the spacetime vector in the rotation plane and leaves components perpendicular to the plane unchanged.

The simple rule for calculating the Lorentz rotation that boosts the frame of a paravector $p = p^0 + \mathbf{p}$ from rest to proper velocity u is

$$p \to p' = up^{\triangle} + p^{\perp} \tag{11}$$

where $p^{\triangle}$ is the part of p that lies coplanar with the rotation plane in spacetime and $p^{\perp}$ is the remaining part, namely the part orthogonal to the rotation plane:

$$p^{\triangle} = p - p^{\perp} = p^0 + (\mathbf{p} \cdot \hat{\mathbf{v}})\ \hat{\mathbf{v}}, \tag{12}$$

where $\hat{\mathbf{v}}$ is the unit vector in the direction of $\mathbf{v}$. The spacetime plane of the Lorentz rotation contains the time axis, $e_0 \equiv 1$, the boost direction $\mathbf{v}$, and all linear combinations of e_0 and $\mathbf{v}$. The part of p in this plane is simply multiplied by u. The part $p^{\perp}$ perpendicular to the rotation plane, with components on the spatial directions perpendicular to $\mathbf{v}$, is unchanged by the transformation.

The transformation (11) is all one needs to boost any spacetime vector. As shown below, its form is the same as for spatial rotations. The inverse transformation is given by replacing u by $\bar{u}$, as seen from (9). No matrices or tensors are needed. To evaluate the boost (11), only the products of scalars and the geometric product of collinear vectors is needed, and as we saw above, the geometric product of collinear vectors is just their scalar product.

In particular, this transformation applies to the basis vectors of paravector space: Suppose our boost is along e_1, that is the proper velocity of the boost is $u = \gamma(1 + ve_1/c)$. Then the transformation

$$e_0 \equiv 1 \to u$$
$$e_1 \to ue_1$$
$$e_2 \to e_2,\ e_3 \to e_3$$

gives the spacetime basis of the frame moving with proper velocity u. This is easily plotted on a spacetime diagram if we note

$$ue_0 = u = \gamma\left(e_0 + \frac{v}{c}e_1\right)$$
$$ue_1 = \gamma\left(e_1 + \frac{v}{c}e_0\right).$$

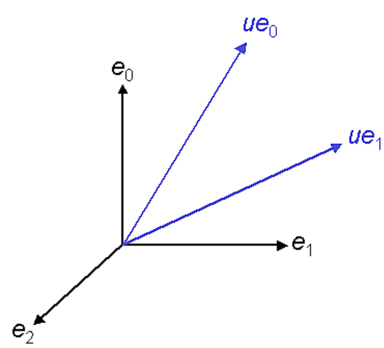

FIG. 3: Boost of the basis for $\mathbf{v} = 0.6c\mathbf{e}_1$.

For example, if $v = 0.6c$, then $\gamma = 5/4$ and the basis paravectors of the moving frame are

$$ue_0 = \frac{5}{4}e_0 + \frac{3}{4}e_1$$
$$ue_1 = \frac{3}{4}e_0 + \frac{5}{4}e_1$$

(see Fig. 3). The perpendicular vectors e_2 and e_3 (not shown) are unchanged.

C. Spacetime Geometry

It is worthwhile to point out some surprising features of spacetime geometry. Note first that the transformed paravectors have the same square length as the original ones:

$$ue_0\left(\overline{ue_0}\right) = u\bar{u} = 1$$
$$ue_1\left(\overline{ue_1}\right) = ue_1\bar{e}_1\bar{u} = -u\bar{u} = -1\ .$$

Obviously the defined square length of spacetime vectors does not correspond to the Euclidean length that would be measured with a ruler on a diagram such as Fig. 3. The Euclidean length of a spacetime vector p is $\langle p^2 \rangle_S^{1/2}$. A good exercise for students is to work out the loci of spacetime vectors of square length ± 1 on a spacetime diagram. The result gives hyperboloids of revolution as seen in Fig. 4, which are asymptotic to the lightcone $r\bar{r} = 0$. Note that one spatial dimension in the e_2e_3 plane has been suppressed in the diagram, and the lightcone, which is drawn as a two-dimensional surface, actually represents the three-dimensional *hypersurface* of a light pulse emitted at the origin.

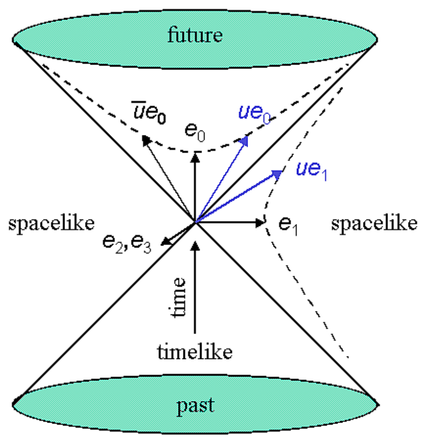

FIG. 4: Geometry of spacetime

All space at an instant in time for an observer at rest is a hypersurface that appears as a horizontal plane in the diagram. The intersection of such a hypersurface with the lightcone gives a circle on the diagram that represents a spherical surface in space. Since Lorentz rotations preserve the square lengths of paravectors, they leave the paravectors within defined regions of spacetime. Spacetime vectors can be classified as either past timelike, future timelike, spacelike, or lightlike (null), and Lorentz rotations do not change the classification.

Note further that the paravectors ue_0 and ue_1 are orthogonal:

$$\langle ue_1 \overline{ue}_0 \rangle_S = \langle \bar{u} u \bar{e}_1 \rangle_S = 0,$$

and a spacetime vector on the lightcone is orthogonal to itself. Orthogonal paravectors are not generally perpendicular (in a Euclidean sense) on the spacetime diagram, but ue_1 is perpendicular to $\overline{ue_0}$. More generally, paravectors p, q are perpendicular (on the diagram) if $\langle pq \rangle_S = 0$, and orthogonal paravectors are always perpendicular to each other's Clifford conjugate.

D. Examples: Lorentz Contraction and Velocity Composition

The time dilation derived above implies Lorentz contraction of a moving object. As an example, consider a car racing at constant velocity $\mathbf{v} = v\mathbf{e}_1$ in the lab. We place a clock at the origin and measure the time for the car to cross the line $\mathbf{x} \cdot \mathbf{e}_1 = 0$. If l is the length of the car in the lab, it takes a time $t = l/v$ for the car to cross. In the frame of the car, the clock has velocity $-\mathbf{v}$ and the crossing time is therefore dilated to $t_0 = \gamma t$, implying a longer car length $l_0 = vt_0 = \gamma l$. The car length in the lab is said to be *Lorentz contracted* by the factor γ^{-1} relative to that in the frame of the car. It is important to specify the motion of the clock that records the events. It is easy for students to forget that the synchronization of clocks at different positions is generally destroyed by a boost. Because boosts are so easily calculated in APS, students can readily work out the full details:

In the lab, the spacetime position of the front of the moving car is $x_1 = ct + \mathbf{v}t$ and the rear of the car is at $x_2 = x_1 - l\mathbf{e}_1$, where l is the length of the car in the lab. The rear of the car crosses the start line when $\mathbf{x}_2 \cdot \mathbf{e}_1 = 0$, that is when $t = l/v$. In the frame of the car, the spacetime positions are found by boosting the car to rest, using the proper velocity $\bar{u}$:

$$\begin{aligned} \bar{u}x_1 &= \bar{u}\left(c + \mathbf{v}\right)t = \gamma^{-1}ct \\ \bar{u}x_2 &= \bar{u}\left(x_1 - l\mathbf{e}_1\right) = \gamma^{-1}ct + \gamma vl/c - \gamma l\mathbf{e}_1 . \end{aligned}$$

The vector part of $\bar{u}\left(x_1 - x_2\right)$ gives the length of the car in its rest frame, namely γl, which is larger than that in the lab by the factor γ.

While the scalar parts, and hence lab times, of x_1, x_2 are equal, the times in the frame of the car differ by $\gamma vl/c^2$. This shows that clocks synchronized in the lab are not synchronized in the rest frame. The scalar part of $\bar{u}x_2$ gives the time at the rear of the car in its rest frame, and by substituting $l = vt$ we get the time of the rear crossing

$$\gamma^{-1}t + \gamma vl/c^2 = \gamma t\left(\gamma^{-2} + v^2/c^2\right) = \gamma t$$

in the car frame, which is just the dilated time of the clock fixed in the lab.

As an even simpler application, consider the composition of two collinear boosts u_{BA} and u_{CB} (u_{BA} can be read as the proper velocity of A with respect to B, etc.). Since the proper velocity u_{BA} is itself a spacetime vector, we can use (11) with u given by u_{CB} to obtain simply the product (note that $\mathbf{u}_{BA}^{\perp} = 0$)

$$u_{CA} = u_{CB} u_{BA} \ . \tag{13}$$

Thus the *addition* of collinear velocities in Galilean transformations becomes a *product* of proper velocities in relativity. The vector part of (13) gives

$$\gamma_{CA} \mathbf{v}_{CA} = \gamma_{CB} \gamma_{BA} \left(\mathbf{v}_{CB} + \mathbf{v}_{BA} \right) \tag{14}$$

and the scalar part is

$$\gamma_{CA} = \gamma_{CB} \gamma_{BA} \left(1 + \mathbf{v}_{CB} \cdot \mathbf{v}_{BA} / c^2 \right) . \tag{15}$$

Their ratio gives the standard result immediately:

$$\mathbf{v}_{CA} = \frac{\mathbf{v}_{CB} + \mathbf{v}_{BA}}{1 + \mathbf{v}_{CB} \cdot \mathbf{v}_{BA} / c^2}, \tag{16}$$

which demonstrates that at speeds small compared to c, the Galilean result $\mathbf{v}_{CA} = \mathbf{v}_{CB} + \mathbf{v}_{BA}$ is obtained.

The composition of non-collinear velocities, while not so commonly given, is easily found from (11):

$$\begin{aligned} u_{CA} &= u_{CB} u_{BA}^{\triangle} + u_{BA}^{\perp} \\ &= \gamma_{CB} \left(1 + \frac{\mathbf{v}_{CB}}{c} \right) \gamma_{BA} \left(1 + \frac{\mathbf{v}_{BA}^{\parallel}}{c} \right) + \gamma_{BA} \frac{\mathbf{v}_{BA}^{\perp}}{c}, \end{aligned}$$

where $\mathbf{v}_{BA}^{\parallel} = \mathbf{v}_{BA} - \mathbf{v}_{BA}^{\perp}$ is the component of $\mathbf{v}_{BA}$ along $\mathbf{v}_{CB}$. The scalar part is as before (15) but the vector part (14) is modified to

$$\gamma_{CA} \mathbf{v}_{CA} = \gamma_{CB} \gamma_{BA} \left(\mathbf{v}_{CB} + \mathbf{v}_{BA}^{\parallel} \right) + \gamma_{BA} \mathbf{v}_{BA}^{\perp},$$

giving

$$\mathbf{v}_{CA} = \frac{\mathbf{v}_{CB} + \mathbf{v}_{BA}^{\parallel} + \mathbf{v}_{BA}^{\perp} / \gamma_{CB}}{1 + \mathbf{v}_{CB} \cdot \mathbf{v}_{BA} / c^2}$$

See also Subsection V.2 on extensions to rotors.

E. Spatial Rotations and Bivectors

To better understand why boosts are considered rotations in spacetime and why they have the algebraic form (11), we compare them to common rotations in physical space. Such spatial rotations are Lorentz rotations in a spatial plane, and they have exactly the same form as (11) but with u replaced by a rotation element such as

$$\cos\theta + \mathbf{e}_2\mathbf{e}_1 \sin\theta = \exp\left(\theta\mathbf{e}_2\mathbf{e}_1\right), \tag{17}$$

which gives a rotation by the angle θ in the plane $\mathbf{e}_2\mathbf{e}_1$, the plane that contains all linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$. The Euler-like relation (17) follows by power-series expansion and the easily verified result that $(\mathbf{e}_2\mathbf{e}_1)^2 = -1$. Any product of orthogonal vectors is called a *bivector* and is an intrinsic representation of the plane spanned by the vector factors. Bivectors generate spatial rotations, and with their help, such rotations can be evaluated with simple algebra instead of matrices. Recall from (2) that perpendicular vectors anticommute. The bivectors $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1$ thus differ by a sign, which may be thought of as indicating the circulation pattern or rotation direction in the plane. Since $(\mathbf{e}_1\mathbf{e}_2)\,\mathbf{e}_1 = -\mathbf{e}_2$ and $(\mathbf{e}_1\mathbf{e}_2)\,\mathbf{e}_2 = \mathbf{e}_1$, the product of $\mathbf{e}_1\mathbf{e}_2$ from the left with any linear combination $\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2$ rotates $\mathbf{v}$ clockwise in the $\mathbf{e}_1\mathbf{e}_2$ plane by 90°. If $\mathbf{e}_1\mathbf{e}_2$ is replaced by its inverse, $\mathbf{e}_2\mathbf{e}_1$, the rotation is counter-clockwise by 90°.

For example, to rotate the paravector

$$p = p^0 e_0 + p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 \tag{18}$$

by the angle θ in the $\mathbf{e}_2\mathbf{e}_1$ plane, we calculate

$$\begin{aligned} p &\to \exp\left(\mathbf{e}_2\mathbf{e}_1\theta\right) p^{\triangle} + p^{\perp} \\ &= \left(\cos\theta + \mathbf{e}_2\mathbf{e}_1\sin\theta\right)\left(p_x\mathbf{e}_1 + p_y\mathbf{e}_2\right) + p^0 e_0 + p_z\mathbf{e}_3. \end{aligned} \tag{19}$$

The form of the transformation (11) ensures that components orthogonal to the plane of rotation are unchanged. Here, components along $\mathbf{e}_0$ and $\mathbf{e}_3$ are invariant while those in the plane are rotated:

$$\begin{aligned} p_x\mathbf{e}_1 + p_y\mathbf{e}_2 &\to \left(p_x\mathbf{e}_1 + p_y\mathbf{e}_2\right)\cos\theta + \left(p_x\mathbf{e}_2 - p_y\mathbf{e}_1\right)\sin\theta \\ &= \left(p_x\cos\theta - p_y\sin\theta\right)\mathbf{e}_1 + \left(p_y\cos\theta + p_x\sin\theta\right)\mathbf{e}_2 \ . \end{aligned}$$

IV. DUALS AND CROSS PRODUCTS

The bivector $\mathbf{e}_1\mathbf{e}_2$ plays a role similar to that of the unit imaginary for rotations in the $\mathbf{e}_1\mathbf{e}_2$ plane. There is a difference, however, in that $\mathbf{e}_1\mathbf{e}_2$ is seen to anticommute with vectors in the plane:

$$\mathbf{e}_1\mathbf{e}_2 p^{\triangle} = -p^{\triangle}\mathbf{e}_1\mathbf{e}_2 \ .$$

It is the *volume element* $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ in APS that can be identified to i : it squares to -1 and commutes with all vectors and their products. With the identification

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i,$$

every bivector in APS is equivalent to an imaginary vector directed perpendicular to the plane of the bivector. For example

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3 = i\mathbf{e}_3.$$

The vector $\mathbf{e}_3$ is said to be *dual* to the plane $\mathbf{e}_1\mathbf{e}_2$. It is common in physics to represent a spatial plane by its dual vector. Indeed, this is the principal use of the vector cross product since the vector dual to a plane is the cross product of vectors in the plane. In the case above, $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. However, the dual vector is an *extrinsic* representation of the plane: it depends on the space in which the plane resides. A two-dimensional plane in a space with four or more dimensions does not possess a unique dual vector. The bivector, as mentioned above, is an *intrinsic* representation of a plane: it depends only on two linearly independent vectors in the plane.

More generally, the bivector part of the product of vectors $\mathbf{p}, \mathbf{q}$ is related to the cross product by

$$i\mathbf{p} \times \mathbf{q} = \frac{1}{2}(\mathbf{pq} - \mathbf{qp}) = \mathbf{pq} - \mathbf{p} \cdot \mathbf{q} \tag{20}$$

The axis of rotation is the vector dual to the rotation plane. Such a vector is called an *axial vector* (or pseudovector), denoting its invariance under such rotations as well as under a spatial inversion of its vector factors. Important examples are the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the magnetic field of a moving charge, both of which are vectors dual to the plane of motion (the plane of $\mathbf{r}$ and $\mathbf{p}$). By using the dual vector, one can write the rotation element $\exp(\mathbf{e}_2\mathbf{e}_1\theta)$ in terms of the rotation axis: $\exp(-i\mathbf{e}_3\theta)$.

A geometrically distinct use of the cross product occurs in the cross product of a vector with a pseudovector, for example $(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$, which arises algebraically as the product of the bivector (20) with the coplanar component $\mathbf{r}^{\triangle}$ of $\mathbf{r}$. As seen above, the result is a vector in the plane of $\mathbf{p}$ and $\mathbf{q}$ that is perpendicular to $\mathbf{r}^{\triangle}$ (and hence to $\mathbf{r}$). A common example of this use of the cross product occurs in the Lorentz force $q\mathbf{v} \times \mathbf{B}$. These two geometrical uses of the cross product can be the source of confusion in usual vector treatments, but they are cleanly distinguished in APS, where vectors and bivectors are distinct elements.

Aside: One can identify $\frac{1}{2}(\mathbf{pq} - \mathbf{qp})$ as the exterior or wedge product $\mathbf{p} \wedge \mathbf{q}$ of the vectors $\mathbf{p}, \mathbf{q}$. This product is important in treatments with differential forms and in geometric algebras of higher dimension. However, its use requires further rules for combining wedge and dot products with other elements, a complication that can be avoided in APS. (See also Section VI.)

A. Boosts as Spacetime Rotations

The relation of spatial rotations to boosts (11) is strengthened by writing the proper velocity of the boost in the explicitly unimodular form

$$u = \exp(w\hat{\mathbf{v}}) = \cosh w + \hat{\mathbf{v}} \sinh w, \tag{21}$$

where the boost parameter w is called the *rapidity* of the boost. A comparison with the defining form (8) establishes that $\gamma = \cosh w$. The unit vector $\hat{\mathbf{v}} = \hat{\mathbf{v}}\bar{e}_0$ represents the spacetime plane of rotation for the boost, namely the plane containing all real linear combinations of the direction $\hat{\mathbf{v}}$ of the boost velocity and the time axis e_0. It is the product of orthogonal directions in spacetime. We discuss spacetime planes more thoroughly below in Section V. The essential difference between boosts and spatial rotations arises from geometrical differences in the planes of rotation. The rotation plane for boosts includes the time axis e_0 and its generator (such as $\hat{\mathbf{v}}\bar{e}_0$) squares to $+1$. Spatial rotations, on the other hand, are generated by a bivector that squares to -1. Because of this difference, the parameter θ for the spatial rotation is periodic, with a rotation by angle $\theta + 2n\pi$ for any integer n giving the same result as one by θ. All possible rotations in the plane are given by a finite range of the parameter, say $0 \le \theta < 2\pi$. However, for boosts, each value of the parameter in the range $-\infty < w < \infty$ gives a distinct boost. Boosts cannot rotate paravectors through the light cone; they can only tilt them within their spacelike or timelike regions (see Fig. 4).

B. Electromagnetic Field

An introductory physics class may also need to transform electromagnetic fields. Under spatial rotations, electric and magnetic fields are transformed exactly like other vectors. However, boosts act differently on electromagnetic fields than on spacetime vectors because the fields transform as spacetime planes rather than spacetime vectors. As above, by "planes" we mean two-dimensional geometrical objects that contain all real linear combinations of two noncollinear paravectors. The boost transformation for spacetime planes can be found from that for paravectors, as illustrated in Fig. 5.

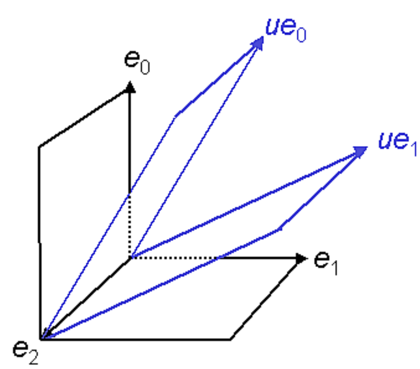

FIG. 5: Boosts of spacetime planes are determined by boosts of the spacetime vectors in them.

The electric and magnetic fields are given by the real and imaginary parts of a covariant electromagnetic field $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$, a complex vector that represents a spacetime plane at a point in spacetime. The real part of $\mathbf{F}$ is the electric field, and its imaginary part is the bivector for a spatial plane whose normal vector is the magnetic field $\mathbf{B}$ (times c in SI units). A static electric field along $\mathbf{e}_2$ sweeps out a spacetime plane containing $\mathbf{e}_2$ and e_0. Under a boost along $\mathbf{e}_1$, this spacetime plane is tilted in the plane of $\mathbf{e}_1$ and e_0, thereby picking up a spatial (horizontal) component in the $\mathbf{e}_1\mathbf{e}_2$ plane that represents a magnetic field in the $\mathbf{e}_3$ direction. Similarly, a magnetic field along $\mathbf{e}_3$, corresponding to a component of the electromagnetic field in the $\mathbf{e}_1\mathbf{e}_2$ plane, is rotated by the boost to give a vertical-plane component corresponding to an electric field along $\mathbf{e}_2$. More generally, under a boost to proper velocity u, $\mathbf{F}$ can be shown to transform as

$$\mathbf{F} \to u\mathbf{F}^{\perp} + \mathbf{F}^{\parallel} \tag{22}$$

where $\mathbf{F}^{\parallel} = \mathbf{F} \cdot \hat{\mathbf{v}}\ \hat{\mathbf{v}} = \mathbf{F} - \mathbf{F}^{\perp}$ is the component of $\mathbf{F}$ along the boost direction. Note that the term $u\mathbf{F}^{\perp}$ is the sum of $\gamma\mathbf{v}\mathbf{E}^{\perp}$, a bivector that contributes to the magnetic field, and $i\gamma\mathbf{v}\mathbf{B}^{\perp} = -i\gamma\mathbf{B}\mathbf{v}^{\triangle}$, a vector that contributes to the electric field. Here $\mathbf{v}^{\triangle}$ is the component of $\mathbf{v}$ in the plane of $i\mathbf{B}$ and hence perpendicular to $\mathbf{B}$, and $-i\gamma\mathbf{B}\mathbf{v}^{\triangle}$ lies in the plane of $i\mathbf{B}$ and is perpendicular to $\mathbf{v}$. The full significance of the spacetime form of the field and a derivation of its transformation can be left for later (see Section V).

This completes a major goal of this paper: to show how arbitrary boosts and spatial rotations can be easily calculated algebraically, without matrices or tensors. Instructors can readily add the many other examples traditionally presented in first courses on relativity, all without matrices or tensors. The formalism maintains close contact with vectors and with the geometry of physical space implicit in the vector notation, and by simply adding time components as scalars to the vectors, it provides an intuitive introduction to relativity suitable for use early in the physics curriculum.

V. EXTENSIONS

APS does considerably more than allow relativistic computations with vectors: it efficiently models the *geometry of spacetime*, representing objects in relativity covariantly as paravectors and their products. We saw above how paravectors in APS unite the space and time components of spacetime vectors. Boosts tilt the paravectors and thereby mix space and time parts.

The material that follows goes beyond what would normally be presented to beginning students, but bright students and instructors may want a fuller understanding of the algebraic formalism as a framework for treating advanced problems in physics. First the APS treatment of planes is extended to spacetime, and then Lorentz transformations are generalized to rotations in arbitrary spacetime planes. Next spin transformations are introduced, and a new decomposition for compound rotations is given. Finally transformations are derived for objects other than spacetime vectors, in particular, for electromagnetic fields, and applied to plane waves to reveal a curious feature of spacetime geometry.

A. Planes in Spacetime

The important geometrical concept of two-dimensional planes in which rotations occur is readily extended to the four dimensions of spacetime. Whereas spatial planes are represented by bivectors, that is, a product of perpendicular vectors, planes in spacetime are represented by *biparavectors,* that is products of two orthogonal paravectors. The plane spanned by orthogonal paravectors p and q is represented by $p\bar{q}$ or $q\bar{p}$. These biparavectors are written as operators that rotate paravectors in the plane into an orthogonal direction in the plane, and they differ by the sense of the rotation. Thus, $p\bar{q}$, when multiplying q from the left, rotates q into the direction $\pm p$ and p into the direction $\pm q$ whereas $q\bar{p} = -p\bar{q}$ rotates in the opposite direction. (The actual signs depend on the signs of $\bar{q}q$ and $\bar{p}p$, respectively.)

There are $\binom{4}{2} = 6$ linearly independent planes in spacetime, and a suitable orthonormal basis of biparavectors is $\{e_1\bar{e}_0, e_2\bar{e}_0, e_3\bar{e}_0, e_3\bar{e}_2, e_1\bar{e}_3, e_2\bar{e}_1\}$. Each basis biparavector generates rotations in a spacetime plane. The first three generate boosts, the last three generate spatial rotations. By using $e_1e_2e_3 = i = e_0\bar{e}_1e_2\bar{e}_3$, we can also express the biparavector basis as $\{e_1, e_2, e_3, ie_1, ie_2, ie_3\}$, which comprises three real and three imaginary vectors in dual pairs. We saw above that imaginary vectors represent spatial planes in terms of their dual (or axial) vectors. The real vectors can be thought of here as *persistent vectors,* ones that persist in time and sweep out timelike planes (planes that contain the time axis $e_0 = 1$) in spacetime. In spacetime, the timelike plane $e_j = e_j\bar{e}_0$ is dual to the spatial plane ie_j. A plane in spacetime can therefore have up to six independent components and be represented by a complex vector.

An important physical plane in spacetime is the electromagnetic field $\mathbf{F}$ at some spacetime point r. If $\mathbf{F}$ is expanded in the biparavector basis, the coefficients give the usual tensor components $F^{\mu\nu}$:

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B} = \frac{1}{2}\sum_{\mu,\nu=0}^{3} F^{\mu\nu}\left\langle e_\mu\bar{e}_\nu\right\rangle_V , \tag{23}$$

where $\left\langle e_\mu\bar{e}_\nu\right\rangle_V = \frac{1}{2}\left(e_\mu\bar{e}_\nu - e_\nu\bar{e}_\mu\right)$ is the vector-like part of $e_\mu\bar{e}_\nu$; it vanishes when $\mu = \nu$ but otherwise equals $e_\mu\bar{e}_\nu$. The expansion (23) relates the tensor components $F^{\mu\nu}$ of the electromagnetic field $\mathbf{F}$ to local vector components of $\mathbf{E}$ and $\mathbf{B}$. The factor of $\frac{1}{2}$ compensates for the appearance of each basis biparavector twice. We have already noted that $\mathbf{B}$ is the vector dual to spatial planes. The vector $\mathbf{E}$ is an example of a vector that persists in time;

it sweeps out the timelike plane $\mathbf{E} = \mathbf{E}\bar{e}_0$ in time. For any observer, $\mathbf{F}$ may have both a timelike projection $\mathbf{E}$ and a spatial one $ic\mathbf{B}$. We should note that although we have spoken of $\mathbf{F}$ as a single spacetime plane, in four dimensions it is also possible for $\mathbf{F}$ to be the sum of two orthogonal planes, in which every paravector in one is orthogonal to every paravector in the other. In contrast to spatial planes in three dimensions, where two planes always share a common vector and can be added to give a single spatial plane, orthogonal planes in four dimensions are dual to each other and cannot be combined into a single real plane. However, since dual spacetime objects in APS are related by the volume element $e_0\bar{e}_1e_2\bar{e}_3 = i$, any sum of spacetime planes in APS can be expressed as a complex scalar times a single plane.

B. Rotors and the Group of Lorentz Rotations

The transformations (11) and (19) allow simple calculations of individual boosts and spatial rotations, but they are not particularly convenient for representing a sequence or group of such transformations. Instead, the general Lorentz rotation of paravector p can be expressed as a *spin transformation*[12]

$$p \to LpL^{\dagger}, \tag{24}$$

where the *Lorentz rotor* can be written $L = \pm\exp(\mathbf{W}/2)$. For a *simple* Lorentz rotation, that is, a rotation in a single spacetime plane, $L = +\exp(\mathbf{W}/2)$, and $\mathbf{W}$ is a biparavector that gives both the plane and the magnitude of the rotation. Compound rotations, that is simultaneous rotations in orthogonal planes, can be expressed as a commuting product of two simple rotations. The dagger † indicates reversion, that is the reversal of the order of all vector factors; it is equivalent to hermitian conjugation for any matrix representation in which the basis vectors are hermitian. It is easily seen that $L^{\dagger} = \exp\left(\mathbf{W}^{\dagger}/2\right)$ and that for spatial rotations, $\mathbf{W}$ is "imaginary", that is $\mathbf{W}^{\dagger} = -\mathbf{W}$, whereas for boosts $\mathbf{W}$ is "real": $\mathbf{W}^{\dagger} = \mathbf{W}$. Since $\bar{\mathbf{W}} = -\mathbf{W}$, L is unimodular: $L\bar{L} = 1$.

We can easily show that the spin transformation (24) reduces to the form (11) and (19) for any simple Lorentz rotation. Let $\mathbf{W}$ be the product $r\bar{s}$ of real orthogonal paravectors. Then, for any paravector $\alpha r + \beta s$ in the plane of $\mathbf{W}$, where α, β are real scalars,

$$\mathbf{W}(\alpha r + \beta s) = (\alpha r + \beta s)\mathbf{W}^{\dagger}, \tag{25}$$

with $\mathbf{W}^\dagger = (r\bar{s})^\dagger = \bar{s}^\dagger r^\dagger = \bar{s}r$. It follows from multiple applications of this relation in the power-series expansion of L that

$$L\left(\alpha r+\beta s\right)L^\dagger = L^2\left(\alpha r+\beta s\right). \tag{26}$$

On the other hand, if q is any paravector orthogonal to both r and s (and hence to the plane containing r and s), then since $q\bar{s} = -s\bar{q}$ and $\bar{q}r = -\bar{r}q$,

$$q\mathbf{W}^\dagger = q\bar{s}r = -s\bar{q}r = s\bar{r}q = -\mathbf{W}q\ . \tag{27}$$

It follows that orthogonal paravectors are invariant under rotations in the plane:

$$LqL^\dagger = L\bar{L}q = q. \tag{28}$$

Consequently, the simple Lorentz rotation (24) of an arbitrary paravector p can always be split into two parts that transform distinctly:

$$p \to L\left(p^\triangle + p^\perp\right)L^\dagger = L^2 p^\triangle + p^\perp. \tag{29}$$

Our previous expressions (11) and (19) of boosts and spatial rotations are seen as special cases of (29). From (29) we can establish an explicit relation for the Lorentz rotor in terms of any non-null paravector p in the plane of rotation and the transformed result $r = LpL^\dagger = L^2p$. Thus, as may be verified by squaring,

$$L = \left(rp^{-1}\right)^{1/2} = \frac{(p+r)\,p^{-1}}{\sqrt{2\left\langle (p+r)\,p^{-1}\right\rangle_S}}. \tag{30}$$

Note that Lp lies along $p+r$, which bisects p and r, and that the square-root factor is required for the normalization $L\bar{L} = 1$.

We can readily extend the result (29) for paravector rotations in single planes to compound rotations, in which rotations are made in a pair of dual planes. Because the biparavectors for the dual planes commute, the Lorentz rotor for any compound rotation can be factored into a pair of simple rotations, one for each of the dual planes: $L = L_1L_2 = L_2L_1$. Similarly, any paravector can be uniquely split into components coplanar with the two planes: $p = p^{\triangle_1} + p^{\triangle_2}$, where $p^{\triangle_1}$ is coplanar with the plane of rotation of L_1 and orthogonal to its dual, that is the plane of rotation of L_2, and *vice versa* for $p^{\triangle_2}$. The compound rotation of p is then easily split as follows:

$$LpL^\dagger = L_1L_2\left(p^{\triangle_1} + p^{\triangle_2}\right)L_2^\dagger L_1^\dagger = L_1^2 p^{\triangle_1} + L_2^2 p^{\triangle_2}. \tag{31}$$

The result (29) for simple Lorentz rotations is the special case of the compound case (31) when the exponent of one of the pair of rotors vanishes.

The forms (24) and (29) of the Lorentz rotations are equivalent, and while (29) is often simpler to evaluate, the spin form (24) leads to powerful spinor methods with the Lorentz group. An important example is the use of the spin form (24) to extend Lorentz rotations to products of paravectors. In particular, any product $p\bar{q}$ of paravectors is seen to transform according to

$$p\bar{q} \to \left(LpL^{\dagger}\right)\left(\bar{L}^{\dagger}\bar{q}\bar{L}\right) = Lp\bar{q}\bar{L}\ . \tag{32}$$

The scalar part of $p\bar{q}$ is thus invariant under Lorentz transformations, and any biparavector such as the electromagnetic field $\mathbf{F}$ transforms as

$$\mathbf{F} \to L\mathbf{F}\bar{L}\ . \tag{33}$$

Using methods analogous to those above, for simple rotors $L = \exp\left(\mathbf{W}/2\right)$ in non-null planes $\mathbf{W}$, one can put this into a form analogous to the paravector rotation (29):

$$\mathbf{F} \to L^2\left(\mathbf{F} - \mathbf{F}^{\mathbf{W}}\right) + \mathbf{F}^{\mathbf{W}} \tag{34}$$

where $\mathbf{F}^{\mathbf{W}} = \left\langle \mathbf{F}\mathbf{W}^{-1}\right\rangle_S \mathbf{W}$ is the projection of $\mathbf{F}$ onto $\mathbf{W}$ and its dual. This reduces to the forms (19) and (22) for rotations and boosts, respectively.

C. Example: Boosts and Scaled Rotations

Propagating plane waves are null fields ($\mathbf{F}^2 = 0$), which have the form

$$\mathbf{F} = \left(1 + \hat{\mathbf{k}}\right)\mathbf{E} = \mathbf{E}\left(1 - \hat{\mathbf{k}}\right), \tag{35}$$

where the electric field $\mathbf{E}$ is perpendicular to the direction $\hat{\mathbf{k}}$ of the energy flow. This form follows easily from Maxwell's equations for source-free space applied to a field $\mathbf{F}$ that is assumed to depend on spacetime location x only through the scalar $s = \langle k\bar{x}\rangle_S$, where k is a constant paravector.[2] It is found that k is null and can thus be written

$$k = \frac{\omega}{c}\left(1 + \hat{\mathbf{k}}\right), \tag{36}$$

where c is the wave speed, $\hat{\mathbf{k}}$ is the propagation direction. If the wave is monochromatic, ω is its frequency and the corresponding photon momentum in spacetime is $\hbar k$. We show here that any boost of such a wave is equivalent to a rotation and dilation (rescaling), and that k and $\mathbf{F}$ are rotated and dilated by the same amount.

A boost of $\mathbf{F}$ and k gives

$$\mathbf{F}' = L\mathbf{F}\bar{L} \tag{37a}$$

$$k' = LkL^{\dagger} \tag{37b}$$

with a rotor

$$L = e^{w\hat{\mathbf{v}}/2} = \cosh\frac{w}{2} + \hat{\mathbf{v}}\sinh\frac{w}{2} = L^{\dagger},$$

where w is the rapidity of the boost and $\hat{\mathbf{v}}$ its direction. Now a curious property of null paravectors such as $1+\hat{\mathbf{k}}$ is that they can "gobble" (or "ungobble") factors of $\hat{\mathbf{k}}$ in what has become known as the *pacwoman property:*[2]

$$\hat{\mathbf{k}}\left(1+\hat{\mathbf{k}}\right) = 1+\hat{\mathbf{k}} = \left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{k}}\ . \tag{38}$$

Consequently

$$L\left(1+\hat{\mathbf{k}}\right) = \left(\cosh\frac{w}{2} + \hat{\mathbf{v}}\hat{\mathbf{k}}\sinh\frac{w}{2}\right)\left(1+\hat{\mathbf{k}}\right), \tag{39}$$

and the factor $\cosh\frac{w}{2} + \hat{\mathbf{v}}\hat{\mathbf{k}}\sinh\frac{w}{2}$ is a scalar plus a bivector, which can always be factored into $e^{\alpha/2}R$, the product of a dilation factor $e^{\alpha/2}$ with a unitary rotor R for a spatial rotation in the plane of $\hat{\mathbf{v}}$ and $\hat{\mathbf{k}}$. In the special case that $\hat{\mathbf{v}}\cdot\hat{\mathbf{k}} = 0$, $\hat{\mathbf{v}}\hat{\mathbf{k}}$ is a spatial bivector and we have

$$\begin{aligned} L\left(1+\hat{\mathbf{k}}\right) &= \frac{\cosh\frac{w}{2}}{\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + \hat{\mathbf{v}}\hat{\mathbf{k}}\sin\frac{\theta}{2}\right)\left(1+\hat{\mathbf{k}}\right) \\ &= e^{\alpha/2}R\left(1+\hat{\mathbf{k}}\right), \end{aligned} \tag{40}$$

where the rotor is

$$R = e^{\theta\hat{\mathbf{v}}\hat{\mathbf{k}}/2}, \tag{41}$$

the dilation factor is

$$e^{\alpha/2} = \frac{\cosh\frac{w}{2}}{\cos\frac{\theta}{2}}, \tag{42}$$

and the rotation angle is given by

$$\tan\frac{\theta}{2} = \tanh\frac{w}{2} \tag{43}$$

and thus by $\cos\theta = 1/\cosh w = \gamma^{-1} = e^{-\alpha}$. Combining these results with their reversion and Clifford conjugation, we find that the boost rotates and dilates both $\mathbf{F}$ and k by the same amount:

$$\mathbf{F}' = e^{\alpha} R\mathbf{F}\bar{R} \tag{44a}$$

$$k' = e^{\alpha} RkR^{\dagger}. \tag{44b}$$

VI. COMPARISON TO STA

The formulation of relativistic physics used above is based on the algebra of physical space (APS). The spacetime algebra (STA) is the geometric algebra based on Minkowski spacetime. It was developed by Hestenes[5], who together with Doran and Lasenby[13] and others have applied it to a wide variety of problems in physics. In this section, we compare APS and STA for introductory treatments of relativity.

APS and STA are closely related. Both are geometric algebras that emphasize the geometric significance of vector products and avoid matrices and tensor elements. The starting or ground space in APS is the space of vectors in three-dimensional physical space, whereas STA is based on four-dimensional vectors in Minkowski spacetime. In APS, spacetime vectors are represented by real paravectors, inhomogeneous elements that are sums of scalars and vectors, whereas in STA spacetime vectors are the homogeneous vectors of the ground space. APS starts with vectors in a Euclidean metric and finds the Minkowski metric as the natural metric of paravector space. In STA, the Minkowski metric is assumed from the outset. In APS all the basis vectors can be taken as real or hermitian. This is not possible in STA, where the metric requires either the timelike basis vector to be real and the spacelike ones to be imaginary or vice versa, depending on the metric signature. It is usual to assume a negative metric signature in STA; the adoption of a Minkowski spacetime metric with positive signature would create a nonequivalent algebra. In APS, the signature of the paravector metric is trivially changed by reversing the sign of the quadratic form for paravectors, that is, what one identifies as their square lengths.

The number of real linearly independent elements in APS is 8, whereas in STA it is 16. Any element in APS can be expressed as a complex paravector, where the four parts, namely the real and imaginary scalar parts and the real and imaginary vector parts represent distinct geometrical entities. APS and STA are related by an isomorphism between APS and the even subalgebra of STA. As a result, anything calculated in APS can also be calculated in STA. Less obvious, it has recently been shown[14] that any measurable physical process that can be represented in STA can also be treated in APS. Both APS and STA give covariant descriptions of relativistic phenomena, but in STA the extra size also admits an absolute-frame model, in which each inertial observer and each field or object frame is absolute, whereas in APS, only the experimentally verifiable relative nature of spacetime vectors and frames is posited. The relation of APS to physical-space vectors is immediate, and different observers in APS see different splits of spacetime vectors into scalar and vector parts as well as different splits of the electromagnetic field into electric and magnetic parts. In STA, spacetime vectors and fields can be absolute, and the measurable components arise by multiplying the absolute object frame by the absolute observer time axis and extracting a space/time split of the resulting bivector.

For introductory treatments, it is especially convenient that the volume element in APS is the unit imaginary scalar, and that it is the same for both physical space and for spacetime. The volume element I in STA, on the other hand, although it is called a pseudoscalar and is used to define dual elements, it actually anticommutes with all vectors and thus acts rather like a fifth dimension. The wedge product in APS is largely avoided, and indeed its definition is problematic for inhomogeneous elements such as paravectors. However, it is used extensively in STA, where one must pay attention to rules for combining contractions (dot products) and wedge products and to the fact that the introduction of I into a product can change one type of product into the other. An extra conjugation, the Clifford conjugation, is required in APS but not STA. However, as seen above in Subsection III-C, the Clifford conjugation provides a simple way to relate lengths and orthogonality in Minkowski spacetime with measured (Euclidean) lengths and perpendicularity on a spacetime diagram.

Some of the efficiency of APS in representing relativistic phenomena arises from the double role playing of its element types. Thus a real scalar may be a Lorentz invariant or it may be the time component of a spacetime vector, and a real vector may be the spatial part of a spacetime vector or the real part of a spacetime plane. Such ambiguity of roles may appear a potential source of confusion, but in fact it is part of our language and common usage. The context usually makes the role clear. For example, a proper-time interval (times c) is both a Lorentz-invariant spacetime displacement and the time component of the displacement in the particle rest frame. Similarly, the mass of a particle (times c) is both the Lorentz invariant length of its spacetime momentum and the time component of the momentum in the particle rest frame. In APS, the same element plays both roles, but in STA, one distinguishes, say, between the Lorentz invariant mc and the time component of the spacetime momentum, $mc\gamma_0$. Similarly, in APS the electric field $\mathbf{E}$ is both a vector and, through its persistence in time, a timelike plane $\mathbf{E}\bar{e}_0$ in spacetime, whereas in STA, $\mathbf{E}$ and $\mathbf{E}\gamma_0$ remain distinct.

APS, in contrast to STA, provides a more direct extension of physical space with fewer new rules and operators. The volume element, through which duality is defined, is the same for three-dimensional physical space as for four-dimensional paravector space, which represents spacetime. It is the true pseudoscalar for both spaces in that it commutes with all elements of the algebra. These distinctions argue for APS as the preferred vehicle for teaching relativity in introductory physics. STA and, for that matter, the differential-forms approach require more overhead and may be more easily approached after students have been exposed to many of the concepts of geometric algebra, and after they have mastered some of the spacetime geometry from APS.

VII. CONCLUSION

Hestenes[15] has advocated geometric algebra as a universal language of physics. The proposal made in the present article is a modest step in that direction. The algebra of physical space, equivalent to the even half of Hestenes' spacetime algebra,[6] offers a simple approach to special relativity that is suitable for introduction at an early stage into the physics curriculum. As its name suggests, APS is based on physical (three-dimensional Euclidean) space that is familiar to beginning students. However, it also includes a four-dimensional linear space of paravectors, which are formed by adding scalar time components to the vectors. The algebra builds on the natural appearance of the Minkowski spacetime metric in paravector space. Paravectors represent spacetime vectors, and their products represent planes and other geometrical structures in spacetime. Rotations of paravectors and their products give the physical Lorentz rotations. Only basic elements of the algebra are needed for the student to easily calculate arbitrary boosts and rotations without recourse to matrices or tensors. Space and time are united in the spacetime continuum of paravector space, but familiar spatial vectors and their physical significance are never left behind.

Other geometric algebras can also be used to formulate relativistic physics, and STA has been especially well developed for this purpose. However, APS is based more immediately on the familiar vectors of physical space, and it requires only half as many independent elements as STA and avoids some of the abstractions and potential hazards such as wedge products and noncommuting pseudoscalars. Even Hestenes seems to agree that APS is the best algebra for introducing relativity, since this is what he has used in the second edition of his mechanics text.[4] A simpler alternative candidate is the algebra of complex quaternions. It has a long history[16–18] as a mathematical framework for relativity, but its broader use in the physics community has remained limited, probably because the geometric interpretation is rather disguised and it does not lend itself easily to a covariant formulation.

The APS approach is part of a comprehensive geometric algebra that nicely displays both spatial and spacetime symmetries and can be applied to all areas of relativistic physics. Much of the power of Clifford's geometric algebras can be traced to their intrinsic representation of planes as bivectors, which can generate rotations. Planes in the paravector space of APS are represented by biparavectors. They not only generate Lorentz transformations but also covariantly represent physical entities such as the electromagnetic field. Some results of APS, such as analytic solutions for the relativistic motion of charges in fields of propagating plane-wave pulses, are easy to obtain in the algebra[19] but almost impossible to find without use of its spinor and projector tools. In this paper, I have illustrated the algebraic language with several simple examples, a couple of new insights into understanding and computing the geometry of spacetime in APS, and by proving that any boost of the electromagnetic field of a propagating plane wave is equivalent to a rotation and a dilation, and that the same rotation and dilation result for both the field and its propagation (and thus momentum) paravector.

The successful introduction of relativity as an integral part of the first-year physics curriculum faces difficulties regardless of the approach taken. The concept of an observer-dependent spacetime in place of absolute space and independent universal time may have a certain intuitive appeal, but since that spacetime has an extra dimension and is no longer Euclidean, it will challenge and stretch the minds of even the best students. Nevertheless, that is the physical world as we have known it for almost a century, and it is high time that we taught it directly to our beginning physics students. This will only be practical if a suitable mathematical approach is used, one based on the geometry of space and spacetime that avoids the unnecessary baggage of tensor and matrix components. Proposing such an approach has been the primary purpose of this paper. The challenge of relativistic concepts may be too great for quantitative calculations in service courses to biologists or engineers, say, but it may also be just the ingredient that excites good students to major in physics. Certainly being able to recognize and use relativistic symmetries will facilitate studies in areas of physics such as electromagnetic theory.

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