GEOMETRY OF PARAVECTOR SPACE, WITH APPLICATIONS TO RELATIVISTIC PHYSICS

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Abstract Clifford's geometric algebra, in particular the algebra of physical space (APS), lubricates the paradigm shifts from the Newtonian worldview to the post-Newtonian theories of relativity and quantum mechanics. APS is an algebra of vectors in physical space, and its linear subspaces include a 4-dimensional space of paravectors (scalars plus vectors). The metric of the latter has the pseudo-Euclidean form of Minkowski spacetime, with which APS facilitates the transition from Newtonian mechanics to relativity without the need of tensors or matrices. APS also provides tools, such as spinors and projectors, for solving classical problems and for smoothing the transition to quantum theory. This lecture concentrates on paravectors and applications to relativity and electromagnetic waves. A following lecture will extend the treatment to the quantum/classical interface.

Keywords: algebra of physical space, Clifford algebra, paravectors, instruction in the quantum age, quantum/classical interface, relativity

Introduction

Devices such as single-electron transistors and switches, single photon masers, and quantum computers are hot topics. Not only is it difficult to ignore quantum and relativistic effects in such devices; these effects may often be dominant. Indeed, the theory of electromagnetic phenomena is inherently relativistic, and impetus behind quantum computation is to employ quantum superposition to make a new breed of computer vastly superior to current ones for certain tasks.

Relativity and quantum theory were introduced roughly a century ago. They both entail paradigm shifts from the assumptions made by Newton, but our teaching has not fully adapted. Undergraduate instruction still begins with Newtonian mechanics, and only after Newton's worldview is ingrained does it progress to the more difficult and abstract mathematics of quantum theory and relativity. In this lecture, I suggest that much of the apparent dichotomy between the mathematics of Newtonian mechanics and that of both quantum theory and relativity arises because we have not been using the best formulation to describe the physics.

Clifford's geometric algebra, in particular the *algebra of physical space (APS),* empowers classical physics with geometric tools that lead to a covariant formulation of relativity and are strikingly similar to tools common in quantum theory.[1, 2] With APS, quantum theory and relativity can be taught with the same mathematics as Newtonian physics, and this permits an earlier, smoother introduction to post-Newtonian physics. This, in turn, encourages students to build intuition consistent with relativistic and quantum phenomena and properly prepares them for the quantum age of the 21st century.

The principal purpose of this lecture is to demonstrate that the structure and geometry of APS makes it a natural and minimal model for both Newtonian and relativistic mechanics. It is natural in that it associates quantities much in the same way that humans usually do, and it is minimal in that it avoids the assumption of additional structure that is not relevant to the physics. I start by reviewing the Clifford algebras commonly used in physics and their relation to APS, which I introduce as an algebra of spatial vectors. However, we quickly note that APS contains a 4-dimensional paravector space with the metric of spacetime, and it can be used to formulate a covariant approach to relativity. Multiparavectors and their Lorentz transformations are also discussed and interpreted. The relation of APS to the spacetime algebra (STA) is discussed in detail with an emphasis on the difference between absolute and relative formulations of relativity. The quantum-like tools of eigenspinors and projectors are introduced, along with applications to the electrodynamics of Maxwell and Lorentz, including a study of Stokes parameters and light polarization.

1. Clifford Algebras in Physics

The importance of Clifford algebras to physics and engineering is increasingly recognized,[3–8] and most physicists have encountered Clifford algebras in some guise. The three most commonly employed in physics are the quaternion algebra $\mathbb{H} = C\ell_{0,2}$, the algebra of physical space (APS) $C\ell_3$, and the spacetime algebra (STA) $C\ell_{1,3}$. They are closely related. We use the common notation[9] $C\ell_{p,q}$ for the Clifford algebra of a vector space of metric signature (p,q) , and $C\ell_p \equiv C\ell_{p,0}$.

When Hamilton introduced vectors in 1843, they were part of an *algebra of quaternions.*[10] The superiority of $\mathbb{H}$ for matrix-free and coordinate-free computations of rotations in physical space has been recently rediscovered by space programs, the computer-games industry, and robotics engineering. Furthermore, $\mathbb{H}$ has been investigated as a replacement of the complex field in an extension of Dirac theory.[11] Quaternions were used by Maxwell and Tait to express Maxwell's equations of electromagnetism in compact form, and they motivated Clifford to find generalizations based on Grassmann theory.

Hamilton's biquaternions (complex quaternions) are isomorphic to APS: $\mathbb{H} \otimes \mathbb{C} \simeq \mathcal{C}\ell_3$, familiar to physicists as the algebra of the Pauli spin matrices. The even subalgebra $\mathcal{C}\ell_3^+$ is isomorphic to $\mathbb{H}$ over the reals, and the correspondences $\mathbf{i} \leftrightarrow \mathbf{e}_3\mathbf{e}_2,\ \mathbf{j} \leftrightarrow \mathbf{e}_1\mathbf{e}_3,\ \mathbf{k} \leftrightarrow \mathbf{e}_2\mathbf{e}_1$ identify pure quaternions with bivectors in APS. APS distinguishes cleanly between vectors and bivectors, in contrast to most approaches with complex quaternions. The identification of the volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i$ (see next section) endows the unit imaginary with geometrical significance and helps explain the widespread use of complex numbers in physics. [12] The sign of i is reversed under parity inversion, and imaginary scalars and vectors correspond to pseudoscalars and pseudovectors, respectively.

APS is also isomorphic to the even part of STA: $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell_{1,3}^+$. STA is familiar as the algebra of Dirac's gamma matrices, where each matrix $\gamma_\mu,\ \mu = 0, 1, 2, 3$, represents a unit vector in spacetime. To be sure, Dirac's electron theory (1928) was based on a matrix representation of $\mathcal{C}\ell_{1,3}$ over the *complex field*, whereas STA, pioneered by Hestenes[13–15] for use in many areas of physics, is $\mathcal{C}\ell_{1,3}$ over the *reals*.

Clifford algebras of higher-dimensional spaces have also been used in robotics[16], many-electron systems, and elementary-particle theory[17]. This lecture focuses on APS, although generalizations to $\mathcal{C}\ell_n$ are made where convenient, and one section is devoted to the relation of APS to STA. A full study of APS is beyond the scope of this lecture and can be found elsewhere[6], but the algebra is sufficiently simple that we can easily present its foundation and structure.

APS: an Algebra of Vectors

To form any algebra, we need elements and an associative product among them. The elements of APS are the vectors of physical space $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and all their products $\mathbf{uv}, \mathbf{uvw}, \mathbf{uu}, \ldots$. If we start with vectors in an n-dimensional Euclidean space, then only one axiom is needed to define the algebraic product: the square of any vector $\mathbf{u}$ is its square length (a real number, a scalar):

$$\mathbf{uu} \equiv \mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u} \ . \tag{1}$$

That's it. This axiom, together with the usual rules for adding and multiplying square matrices, determines the entire algebra.

Let's put $\mathbf{u} = \mathbf{v} + \mathbf{w}$. The axiom implies that

$$\mathbf{vw} + \mathbf{wv} = 2\mathbf{v} \cdot \mathbf{w} \ . \tag{2}$$

Evidently the algebra is not commutative. If $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, they anticommute. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots\}$ be a basis of orthogonal unit vectors in the n-dimensional Euclidean space. Then $\mathbf{e}_1^2 = 1$ and $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$. We can be sure that $\mathbf{e}_1\mathbf{e}_2$ doesn't vanish because it squares to -1 : $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -1$.The product of perpendicular vectors is a new element called a *bivector*.It represents a directed area in the plane of the vectors. The "direction" corresponds to circulation in the plane: if the circulation is reversed, the sign of the bivector is reversed. The bivector replaces the vector cross product of polar vectors, but unlike the usual cross product, it is intrinsic to the plane and can be applied to planes in spaces of more than 3 dimensions.

Bivectors can also be viewed as operators on vectors. They generate rotations and reflections in the plane. To rotate any vector $\mathbf{u} = u^1\mathbf{e}_1 + u^2\mathbf{e}_2$ in the $\mathbf{e}_1\mathbf{e}_2$ plane by a right angle, multiply it by the unit bivector $\mathbf{e}_1\mathbf{e}_2 : \mathbf{u}\mathbf{e}_1\mathbf{e}_2 = u^1\mathbf{e}_2 - u^2\mathbf{e}_1$. The counterclockwise sense of the rotation when $\mathbf{u}$ is multiplied from the right corresponds to the circulation used to define the "direction" of $\mathbf{e}_1\mathbf{e}_2$. Multiplication from the left reverses the rotation. To rotate $\mathbf{u}$ in the plane by an arbitrary angle ϕ multiply it by a linear combination of 1 (no rotation) and $\mathbf{e}_1\mathbf{e}_2$:

$$\mathbf{u}\,(\cos\phi + \mathbf{e}_1\mathbf{e}_2 \sin\phi) = \mathbf{u} \exp\left(\mathbf{e}_1\mathbf{e}_2\phi\right)$$

Note the exponential function of the bivector $\mathbf{e}_1\mathbf{e}_2\phi$. It can be defined by its power-series expansion because all powers of bivectors can be calculated in the algebra. The Euler-type relation for $\exp\left(\mathbf{e}_1\mathbf{e}_2\phi\right)$ follows from the fact that $\mathbf{e}_1\mathbf{e}_2$ squares to -1.

A general vector $\mathbf{v}$, with components both in the plane and perpendicular to it, is rotated by the angle ϕ in the $\mathbf{e}_1\mathbf{e}_2$ plane by

$$\mathbf{v} \to R\mathbf{v}R^\dagger, \tag{3}$$

where the *rotors* $R, R^\dagger$ are

$$R = \exp\left(-\mathbf{e}_1\mathbf{e}_2\phi/2\right) = \cos\frac{\phi}{2} - \mathbf{e}_1\mathbf{e}_2 \sin\frac{\phi}{2} \tag{4}$$

$$R^\dagger = \cos\frac{\phi}{2} - (\mathbf{e}_1\mathbf{e}_2)^\dagger \sin\frac{\phi}{2} = \cos\frac{\phi}{2} - \mathbf{e}_2\mathbf{e}_1 \sin\frac{\phi}{2} = R^{-1} \ . \tag{5}$$

The dagger † denotes a conjugation called *reversion,*[1] in which the order of vectors in products is reversed. Thus, for any vectors $\mathbf{v}, \mathbf{w},$, $(\mathbf{vw})^\dagger = \mathbf{wv}$.The reversion of other elements, say $\mathbf{AB}$, can then be found from the rule $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$. An element equal to its reversion is said to be *real*, whereas one equal to minus its reversion is *imaginary*. The two-sided *spinorial form* of (3) preserves the reality of the transformed vector. From (5), R is unitary and consequently all products of vectors transform in the same way (3). In particular, the bivector $\mathbf{e}_1\mathbf{e}_2$ commutes with the rotors $\exp(\pm\mathbf{e}_1\mathbf{e}_2\phi/2)$ and is therefore invariant under rotations in the $\mathbf{e}_1\mathbf{e}_2$ plane. It is equally well expressed as the ordered product of any pair of orthonormal vectors in the plane.

The trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ squares to -1 and commutes with all vectors that are linear combinations of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$. More generally, products of k orthonormal basis vectors $\mathbf{e}_j$ can be reduced if two of them are the same, but if they are all distinct, their product is a basis k-vector. In an n-dimensional space, the algebra contains $\binom{n}{k}$ such linearly independent k-vectors, and any real linear combination of them is said to be an element of *grade* k . Thus, scalars have grade 0, vectors grade 1, bivectors grade 2, trivectors grade 3, and so on.

In APS, where the number of dimensions is $n = 3$, the $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the highest-grade element, namely the *volume element*, and it commutes with every vector and hence with all elements. It can be identified with the unit imaginary:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i \ . \tag{6}$$

Note that i changes sign under spatial inversion: $\mathbf{e}_k \rightarrow -\mathbf{e}_k,\ k = 1, 2, 3$. Imaginary scalars are called *pseudoscalars* because of this sign change. Any bivector can be expressed as an imaginary vector, called a *pseudovector*. For example,

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\left(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right)/i = i\mathbf{e}_3. \tag{7}$$

The center of APS (the part that commutes with all elements) is spanned by $\{1, i\}$ and is identified with the complex field. Every element of APS is a linear combination of $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, i\mathbf{e}_1, i\mathbf{e}_2, i\mathbf{e}_3, i$ over the reals or, equivalently, of $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ over the complex field. The element $^*x = -ix$ is said to be the *Clifford-Hodge dual*of x.

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[1] A tilde ˜is often used to indicate reversal, but in spaces of definite metric such as Euclidean spaces, the dagger is common since it corresponds to Hermitian conjugation in any matrix representation in which the matrices representing the basis vectors are Hermitian.

Existence

We should check that our algebra exists. It is possible to define structures that are not self-consistent. The existence of a *matrix representation* is sufficient to prove existence. The canonical one replaces unit vectors by Pauli spin matrices. There are an infinite number of valid representations. They share the same *algebra* and that is all that matters.

2. Paravector Space as Spacetime

APS includes not only the 3-dimensional linear space of physical vectors, but also other linear spaces, in particular the 4-dimensional linear space of scalars plus vectors. In mathematical terms, this 4-dimensional linear space is also a vector space and its elements are vectors, but to distinguish them from spatial vectors, we call them *paravectors.* A paravector p can generally be written

$$p = p^0 + \mathbf{p} \tag{8}$$

where p^0 is a scalar and $\mathbf{p} = p^k \mathbf{e}_k,\ k = 1, 2, 3$, a physical vector (the summation convention for repeated indices is used). It is convenient to put $\mathbf{e}_0 = 1$ so that we can write

$$p = p^\mu \mathbf{e}_\mu,\ \mu = 0, 1, 2, 3. \tag{9}$$

The *metric* of the 4-dimensional paravector space is determined by the quadratic form ("square length") of paravectors. Since p^2 is not generally a pure scalar, we need the *Clifford conjugate* of p, $\bar{p} = p^\mu \bar{\mathbf{e}}_\mu = p^0 - \mathbf{p}$ since the product $p\bar{p} = \bar{p}p = \left(p^0\right)^2 - \mathbf{p}^2$ is always a scalar. If we take $p\bar{p}$ to be the quadratic form of p and use

$$\langle x \rangle_S \equiv \frac{1}{2}\left(x + \bar{x}\right) = \langle \bar{x} \rangle_S \tag{10}$$

to denote the scalarlike (that is scalar plus pseudoscalar) part of any element x, then

$$p\bar{p} = \langle p\bar{p} \rangle_S = p^\mu p^\nu \eta_{\mu\nu} \tag{11}$$

and the *Minkowski metric* of spacetime $\eta_{\mu\nu} = \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_S$ arises automatically. Spacetime can be viewed as paravector space, and spacetime vectors are (real) paravectors in APS. The vector part of the paravector is the usual spatial vector, and the scalar part is the time component.

EXAMPLE 1 *In units with the speed of light $c = 1$, the quadratic form (or "square length") of the spacetime displacement $dr = dt + d\mathbf{r}$ is*

$$dr\ d\bar{r} = dt^2 - d\mathbf{r}^2 = dt^2\left(1 - \mathbf{v}^2\right),$$

where $\mathbf{v}$ *is the velocity* $d\mathbf{r}/dt$*. If we define the dimensionless scalar known as the proper time* τ *by* $dr\, d\bar{r} = d\tau^2$*, then we see that* $\gamma d\tau = dt$ *with*

$$\gamma = \frac{dt}{d\tau} = \left[1 - \mathbf{v}^2\right]^{-1/2}.$$

In particular, $d\tau = dt$ *in a rest frame of the displacement. The dimensionless proper velocity is*

$$u = \frac{dr}{d\tau} = \frac{dt}{d\tau}\left(1 + \frac{d\mathbf{r}}{dt}\right) = \gamma\left(1 + \mathbf{v}\right),$$

and by definition it is unimodular: $u\bar{u} = 1$*. Other spacetime vectors can be similarly represented as paravectors in APS. For example, the energy-momentum paravector of a particle is* $p = mu = E + \mathbf{p}$ *.*

In his article[18] of 1905 on special relativity, Einstein did not mention a spacetime continuum. That was a construction proposed three years later by Minkowski. I like to think that Einstein, had he seen it, would have appreciated the natural appearance of the spacetime geometry in paravector space.

Multiparavectors

From the quadratic form of the sum $p + q$ of paravectors, we obtain an expression for the *scalar product* of paravectors p and q :

$$\langle p\bar{q}\rangle_S = \frac{1}{2}\left(p\bar{q} + q\bar{p}\right). \tag{12}$$

One says that paravectors p and q are *orthogonal* if and only if $\langle p\bar{q}\rangle_S = 0$. The *vectorlike* (vector plus pseudovector) part of $p\bar{q}$ is

$$\langle p\bar{q}\rangle_V \equiv \frac{1}{2}\left(p\bar{q} - q\bar{p}\right) = p\bar{q} - \langle p\bar{q}\rangle_S \tag{13}$$

and represents the directed plane in paravector space that contains paravectors p and q. It is called a *biparavector* and can be expanded in a basis of unit biparavectors $\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V$ with $\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V^2 = \pm 1$:

$$\langle p\bar{q}\rangle_V = p^\mu q^\nu \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V$$

The biparavectors form a six-dimensional linear subspace of APS equal to the direct sum of vector and bivector spaces. Since bivectors of APS are also pseudovectors, any biparavector is also a complex vector.

Biparavectors arise most frequently in APS as operators on paravectors. Thus, the unit biparavector $\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu\rangle_V$, $\mu \neq \nu$, rotates any paravector

$p = a\mathbf{e}_\mu + b\mathbf{e}_\nu$ in the plane to an orthogonal direction:

$$\begin{aligned}\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_V p &= \frac{1}{2}\left(a\mathbf{e}_\mu \bar{\mathbf{e}}_\nu \mathbf{e}_\mu + b\mathbf{e}_\mu \bar{\mathbf{e}}_\nu \mathbf{e}_\nu - a\mathbf{e}_\nu \bar{\mathbf{e}}_\mu \mathbf{e}_\mu - b\mathbf{e}_\nu \bar{\mathbf{e}}_\mu \mathbf{e}_\nu\right) \\ &= -a\eta_{\mu\mu}\mathbf{e}_\nu + b\eta_{\nu\nu}\mathbf{e}_\mu .\end{aligned}$$

In analogy with bivectors, biparavectors generate rotations in paravector space. One of the most important biparavectors in physics is the electromagnetic field (or "Faraday") which in SI units with $c = 1$ can be written $\mathbf{F} = \langle \partial \bar{A} \rangle_V = \mathbf{E} + i\mathbf{B}$,where $\partial = \mathbf{e}^\mu \partial / \partial x^\mu$ is the paravector gradient operator and $A = \phi + \mathbf{A}$ is the paravector potential. We will see below how to define $\mathbf{F}$ in terms of rotation rate.

Triparavectors can also be formed. The triparavector subspace of APS is four-dimensional, the direct sum of pseudovector and pseudoscalar spaces. The volume element of paravector space is the same as that of the underlying vector space:

$$\mathbf{e}_0 \bar{\mathbf{e}}_1 \mathbf{e}_2 \bar{\mathbf{e}}_3 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = i \ . \tag{14}$$

As seen before, it commutes with all elements of APS.

Paravector Rotations and Lorentz Transformations

Rotations and reflections in paravector space preserve the scalar product $\langle p\bar{q} \rangle_S$ of any two paravectors. Paravector rotations have the same spinorial form as vector rotations,

$$p \to LpL^\dagger , \tag{15}$$

where L is a unimodular element ($L\bar{L} = 1$) known as a *Lorentz rotor*. Lorentz rotations are the physical Lorentz transformations of relativity: boosts, spatial rotations, and their products. In APS they can be calculated algebraically without matrices or tensors.

Lorentz rotors for spatial rotations are just the same rotors (4) introduced above, and those for boosts are similar except that the rotation plane in paravector space includes the time axis $\mathbf{e}_0$. For a boost along $\mathbf{e}_1$ for example, the Lorentz rotor L has the real form

$$L = \exp\left(\mathbf{e}_1 \bar{\mathbf{e}}_0 \frac{w}{2}\right) = \cosh\frac{w}{2} + \mathbf{e}_1 \sinh\frac{w}{2} = L^\dagger , \tag{16}$$

where w is a scalar parameter called the *rapidity*. The Lorentz rotation (15) can be calculated directly with two algebraic products, but because it is linear in p, it is sufficient to determine the transformed basis paravectors

$$\mathbf{u}_\mu \equiv L\mathbf{e}_\mu L^\dagger . \tag{17}$$

Since the proper velocity in the rest frame is unity, the Lorentz rotation of $\mathbf{e}_0 = 1$) must give the proper velocity u induced by the boost on objects initially at rest:

$$u = L\mathbf{e}_0 L^\dagger = LL^\dagger = \gamma\left(1 + v\mathbf{e}_1\right) \tag{18}$$

$$\gamma = \cosh w,\ \gamma v = \sinh w\ . \tag{19}$$

Since $\mathbf{e}_1$ also commutes with the biparavector $\mathbf{e}_1\bar{\mathbf{e}}_0$ of the rotation plane whereas $\mathbf{e}_2$ and $\mathbf{e}_3$ anticommute with it, the boost of the paravector basis elements gives

$$\begin{aligned} \mathbf{u}_0 &= u\mathbf{e}_0,\ \mathbf{u}_1 = u\mathbf{e}_1 \\ \mathbf{u}_2 &= \mathbf{e}_2,\ \mathbf{u}_3 = \mathbf{e}_3\ . \end{aligned} \tag{20}$$

The set $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \equiv \{\mathbf{u}_\mu\}$ is an orthonormal basis of paravectors for the boosted system. It follows that the boost of any paravector $p = p^\mu\mathbf{e}_\mu$ produces

$$\begin{aligned} LpL^\dagger &= p^\mu\mathbf{u}_\mu = u\left(p^0\mathbf{e}_0 + p^1\mathbf{e}_1\right) + p^2\mathbf{e}_2 + p^3\mathbf{e}_3 \\ &= \gamma\left(p^0 + vp^1\right)\mathbf{e}_0 + \gamma\left(p^1 + vp^0\right)\mathbf{e}_1 + p^2\mathbf{e}_2 + p^3\mathbf{e}_3\ . \end{aligned} \tag{21}$$

We can eliminate the dependence on the paravector basis by introducing components of p coplanar with the rotation plane

$$p^\triangle = p^0\mathbf{e}_0 + p^1\mathbf{e}_1$$

and perpendicular to it

$$p^\perp = p^2\mathbf{e}_2 + p^3\mathbf{e}_3 = p - p^\triangle.$$

Then the boost of p is

$$p \to LpL^\dagger = up^\triangle + p^\perp. \tag{22}$$

No matrices or tensors are required, and the algebra is trivial. Algebraic calculation of the boost is sufficiently simple to be taught at an early stage of a student's study of physics. Students can perform most calculations in introductory relativity texts using no more than the transformation (22) and the basic axiom (2) of the algebra. A couple of examples will illustrate the simplicity of the approach.

EXAMPLE 2 *Let Carol have proper velocity u_{BC} as seen by Bob, and let Bob have proper velocity u_{AB} with respect to Alice. To find Carol's proper velocity u_{AC} with respect to Alice, we boost the paravector $p = u_{BC}$ by $L = L^\dagger = u_{AB}^{1/2}$ as in (22). If the vector parts of u_{AB} and u_{BC} are* *collinear, u_{AB} and u_{BC} lie in the same spacetime plane and commute. Then $p^{\perp} = 0$ and the transformation (22) reduces to the product*

$$u_{AC} = u_{AB} u_{BC} \ .$$

By writing each proper velocity in the form $u = \gamma(1 + \mathbf{v})$, one easily extracts the usual result for collinear velocity composition[2]

$$\mathbf{v}_{AC} = \frac{\langle u_{AC} \rangle_V}{\langle u_{AC} \rangle_S} = \frac{\mathbf{v}_{AB} + \mathbf{v}_{BC}}{1 + \mathbf{v}_{AB} \cdot \mathbf{v}_{BC}}.$$

Example 3 *Consider a the change in the wave paravector k of a photon when its source is boosted from rest to proper velocity u. The wave paravector, like the momentum $\hbar k$, is null, $k\bar{k} = 0$, and can be written $k = \omega + \mathbf{k} = \omega\left(1 + \hat{\mathbf{k}}\right)$, where the unit vector $\hat{\mathbf{k}}$ gives the direction and ω the frequency of k. From (22), writing out the coplanar part of k as $k^{\triangle} = \omega + \mathbf{k}^{\|}$ and noting that $k^{\perp}$ is a vector, we find*

$$k = \omega\left(1 + \hat{\mathbf{k}}\right) \to k' = LkL^{\dagger} = u\left(\omega + \mathbf{k}^{\|}\right) + \mathbf{k}^{\perp}$$

with $u = \gamma(1 + \mathbf{v})$ and $\mathbf{k}^{\|} = \mathbf{k} \cdot \hat{\mathbf{v}}\ \hat{\mathbf{v}} = \mathbf{k} - \mathbf{k}^{\perp}$, where $\hat{\mathbf{v}}$ is the unit vector along $\mathbf{v}$. This transformation describes what happens to the photon momentum when the light source is boosted. Evidently $\mathbf{k}^{\perp}$ is unchanged, but there is a Doppler shift in ω and a change in $\mathbf{k} \cdot \hat{\mathbf{v}}$:

$$\begin{aligned} \omega' &= \left\langle u\left(\omega + \mathbf{k}^{\|}\right)\right\rangle_S = \gamma\omega\left(1 + \hat{\mathbf{k}} \cdot \mathbf{v}\right) \\ \mathbf{k}' \cdot \hat{\mathbf{v}} &= \left\langle u\hat{\mathbf{v}}\left(\omega + \mathbf{k}^{\|}\right)\right\rangle_S = \gamma\omega\left(v + \hat{\mathbf{k}} \cdot \hat{\mathbf{v}}\right) = \omega' \cos\theta'. \end{aligned}$$

The ratio $\mathbf{k}' \cdot \hat{\mathbf{v}}/\omega'$ shows how the photons are thrown forward

$$\cos\theta' = \frac{v + \cos\theta}{1 + v\cos\theta} \ .$$

in what is called the "headlight" effect

Example 4 *The results of the previous example can be combined with a qualitative description of Thomson scattering to explain how high-energy gamma-ray photons are produced near sources of energetic electrons. In Thomson scattering, a electron initially at rest scatters an unpolarized*

[2] Note that the cleanest way to compose general Lorentz rotations is to multiply rotors: $L_{AC} = L_{AB} L_{BC}$. See also the section below on eigenspinors.

beam of radiation into all directions. It is the limit of Compton scattering when $\omega \ll m$, the rest energy of the electron. In the Lab frame, the electrons are ultra-relativistic with energies $\gamma m \gg m$, and they collide with photons of the $2.7\,\mathrm{K}$ blackbody radiation that permeates space. Let ω_0 be Lab frequency of the background radiation. In the rest frame of the electron, this gets Doppler shifted to roughly $\gamma\omega_0$ (within a factor between 0 and 2, depending on angle), and Thomson scattering occurs, redistributing the photons in all directions. Transforming the scattered photons back to the Lab gives a collimated beam of photons in the direction of the electron velocity with energies of order $\gamma^2\omega_0$. Thus, $5\,\mathrm{GeV}$ electrons ($\gamma = 10^4$) can raise $10^{-2}\,\mathrm{eV}$ photons to MeV energies.

An attractive feature of this simple algebraic approach to introductory relativity is that it is not restricted to such simple cases. Rather it is part of an algebra that simplifies computations for all relativistic phenomena. Lorentz rotations of multiparavectors, in particular, are readily found by putting together those for paravectors. Thus, for the boost along $\mathbf{e}_1$ considered above, the biparavectors $\mathbf{e}_1\bar{\mathbf{e}}_0$ and $\mathbf{e}_2\bar{\mathbf{e}}_3$ are seen to be invariant whereas $\mathbf{e}_1\bar{\mathbf{e}}_2, \mathbf{e}_1\bar{\mathbf{e}}_3, \mathbf{e}_0\bar{\mathbf{e}}_2, \mathbf{e}_0\bar{\mathbf{e}}_3$ are multiplied from the left by u. More generally, the paravector product $p\bar{q}$ transforms to $LpL^\dagger\overline{(LqL^\dagger)} = Lp\bar{q}\bar{L}$. From this we can confirm that scalar products (12) of paravectors are Lorentz invariant and that any biparavector, say $\mathbf{F}$, transforms as

$$\mathbf{F} \to L\mathbf{F}\bar{L}\ . \tag{23}$$

The power of APS allows us to generalize expression (22) to an arbitrary Lorentz rotation. We start with an arbitrary *simple Lorentz rotation*, that is, a rotation in a single paravector plane. Simple rotations include all spatial rotations, pure boosts, and many boost-rotation combinations. We again split p into one component $p^\triangle$ coplanar with the rotation plane plus another component $p^\perp$ perpendicular to it: $p = p^\triangle + p^\perp$. Consider a biparavector $\langle u\bar{v}\rangle_V$, which represents the plane containing the independent paravectors u and v. By expansion it is easy to see that

$$\langle u\bar{v}\rangle_V u = \frac{u\bar{v}u - v\bar{u}u}{2} = u\left(\frac{\bar{v}u - \bar{u}v}{2}\right) = u\,\langle u\bar{v}\rangle_V^\dagger\ .$$

Similarly, one finds $\langle u\bar{v}\rangle_V v = v\,\langle u\bar{v}\rangle_V^\dagger$ .The component $p^\triangle$ coplanar with $\langle u\bar{v}\rangle_V$ is a linear combination of u and v and therefore obeys the same relation

$$\langle u\bar{v}\rangle_V p^\triangle = p^\triangle\,\langle u\bar{v}\rangle_V^\dagger\ .$$

On the other hand, the component $p^{\perp}$ is orthogonal to u and v, $\left\langle \bar{u}p^{\perp}\right\rangle_S = 0 = \left\langle u\bar{p}^{\perp}\right\rangle_S$ and similarly for u replaced by v. Consequently,

$$\langle u\bar{v}\rangle_V \, p^{\perp} = \frac{u\bar{v}p^{\perp} - v\bar{u}p^{\perp}}{2} = -p^{\perp} \, \langle u\bar{v}\rangle_V^{\dagger} \, .$$

It follows that if L is any Lorentz rotation in the $\langle u\bar{v}\rangle_V$ plane, then

$$p^{\triangle} L^{\dagger} = L p^{\triangle}, \;\; p^{\perp} L^{\dagger} = \bar{L} p^{\perp},$$

and the rotation (15) reduces to

$$p \rightarrow L p^{\triangle} L^{\dagger} + L p^{\perp} L^{\dagger} = L^2 p^{\triangle} + p^{\perp}. \tag{24}$$

Expression (22) is a special case of this relation for a pure boost. We can now generalize the result further to include compound Lorentz rotations. Any rotor can be expressed as

$$L = \pm \exp\left(\frac{1}{2}\mathbf{W}\right), \tag{25}$$

and the arbitrary biparavector $\mathbf{W}$ can always be expanded as a sum of simple biparavectors $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 = (1 + i\alpha)\,\mathbf{W}_1$, where α is a real scalar. The relation $\mathbf{W}_2 = i\alpha\mathbf{W}_1$ means that $\mathbf{W}_2$ is proportional to the dual of $\mathbf{W}_1$ so that $\mathbf{W}_1$ and $\mathbf{W}_2$ represent orthogonal planes: every paravector in $\mathbf{W}_1$ is orthogonal to every paravector in $\mathbf{W}_2$. Since $\mathbf{W}_1$ and $\mathbf{W}_2$ commute, L can be written as a product of commuting simple Lorentz rotations:

$$L = L_1 L_2, \;\; L_1 = e^{\mathbf{W}_1/2}, \;\; L_2 = e^{\mathbf{W}_2/2}.$$

Every paravector p can be split into one component, p_1, coplanar with $\mathbf{W}_1$ and another, p_2, coplanar with $\mathbf{W}_2$. Two applications of the simple transformation result (24) gives its generalization

$$p \rightarrow L p L^{\dagger} = L_1 L_2 \left(p_1 + p_2\right) L_2^{\dagger} L_1^{\dagger} = L_1^2 p_1 + L_2^2 p_2 \; . \tag{26}$$

Note that the Lorentz transformations of paravectors p and $p^{\dagger}$ are the same, as are those of biparavectors $\mathbf{F}$ and $\bar{\mathbf{F}}$. However, p and $\bar{p}$ have distinct transformations as do $\mathbf{F}$ and $\mathbf{F}^{\dagger}$. As a consequence, Lorentz transformations mix time and space components of p, but not its real and imaginary parts, whereas such transformations of $\mathbf{F}$ leave $\mathbf{F}$ vectorlike but mix vector (real) and bivector (imaginary) parts.

3. Interpretation

Note that in APS, the Lorentz rotation acts directly on paravectors and their products, and not on scalar coefficients. Contrast this with matrix or tensor formulations where transformations are given only for the coefficients. Of course, the two approaches are easily related. For example, in the boost (21) of p, if p and the transformed $p' = LpL^\dagger$ are expanded in the same basis $\{\mathbf{e}_\mu\}$, we can simply read off the transformation of coefficients:

$$\begin{aligned} p'^0 &= \gamma\left(p^0 + vp^1\right) \\ p'^1 &= \gamma\left(p^1 + vp^0\right) \\ p'^2 &= p^2,\ p'^3 = p^3, \end{aligned}$$

and this is easily cast into standard matrix form. The two approaches are therefore equivalent, but the APS approach of transforming paravectors is more geometric and does not require Lorentz transformations that are aligned with basis elements. Indeed, the algebraic transformation $p \to p' = LpL^\dagger$ is independent of basis.

A basis is needed only to compare measured coefficients. In general, the transformed paravector p' is $p' = p'^\mu \mathbf{e}_\mu = p^\nu \mathbf{u}_\nu$, where as above $\mathbf{u}_\mu = L\mathbf{e}_\mu L^\dagger$ are the transformed basis paravectors. To isolate individual coefficients from an expansion, we introduce reciprocal basis paravectors $\mathbf{e}^\mu$ defined by the scalar products

$$\left\langle \mathbf{e}_\mu \bar{\mathbf{e}}^\nu \right\rangle_S = \delta_\mu^\nu \ ,$$

where δ_μ^ν is the usual Kronecker delta. The reciprocal paravectors of the standard basis elements $\mathbf{e}_\mu$ are $\mathbf{e}^0 = \mathbf{e}_0, \mathbf{e}^k = -\mathbf{e}_k,\ k = 1,2,3.$ With their help, we obtain

$$p'^\mu = p^\nu \left\langle \mathbf{u}_\nu \bar{\mathbf{e}}^\mu \right\rangle_S = p^\nu \left\langle L\mathbf{e}_\nu L^\dagger \bar{\mathbf{e}}^\mu \right\rangle_S \equiv \mathcal{L}^\mu_\nu p^\nu. \qquad (27)$$

Active, Passive, and Relative Transformations

It is common to distinguish active transformations from passive ones. In active transformations, a single observer compares objects in different inertial frames, whereas in passive transformations a single object is observed by inertial observers in relative motion to each other. The transformations described above were active. However, APS accommodates both active and passive interpretations. The mathematics is the same, as seen from expression (27) for the transformation elements

$$\mathcal{L}^\mu_\nu = \left\langle \left(L\mathbf{e}_\nu L^\dagger\right) \bar{\mathbf{e}}^\mu \right\rangle_S = \left\langle \mathbf{e}_\nu \left(L^\dagger \bar{\mathbf{e}}^\mu L\right) \right\rangle_S = \left\langle \mathbf{e}_\nu \overline{\left(\bar{L}\mathbf{e}^\mu \bar{L}^\dagger\right)} \right\rangle_S \ . \qquad (28)$$

Here we used the property that the scalar part of a product is independent of the order $\langle xy\rangle_S = \langle yx\rangle_S$ for any elements x, y of APS. The first equality of (28) finds the components of the transformed basis paravectors on the original basis, whereas the last finds components of the original basis on the inversely transformed basis. These are alternative ways to interpret the same expression.

To see the relation between active and passive transformations more explicitly, consider the passive transformation of a fixed paravector from one observer, say Alice, to another, say Bob. Let $p_A = p_A^\mu \mathbf{e}_\mu$ be the paravector as seen by Alice in terms of Alice's standard (rest) basis. Bob, who moves at proper velocity u with respect to Alice, will measure a different paravector, namely $p_B = p_B^\mu \mathbf{e}_\mu$, with respect to his rest frame. Note that the paravector basis in expansions of both p_A and p_B is the same, namely the standard basis $\{\mathbf{e}_\mu\}$, even though p_A is expressed relative to Alice and p_B relative to Bob. The reason is that the standard basis $\{\mathbf{e}_\mu\}$ is *relative*; it is at rest relative to the observer. To relate p_A and p_B, both must be expressed relative to the same observer. Bob's frame as seen by Alice is

$$\{\mathbf{u}_\mu\} = \left\{L\mathbf{e}_\mu L^\dagger\right\} \tag{29}$$

with $\mathbf{u}_0 = u$. Thus p_A can be written

$$p_A = p_A^\mu \mathbf{e}_\mu = p_B^\nu \mathbf{u}_\nu = L p_B L^\dagger \ , \tag{30}$$

which can be inverted to give the passive transformation

$$p_A \rightarrow p_B = \bar{L} p_A \bar{L}^\dagger. \tag{31}$$

This has the same form as the active transformation, but with the inverse Lorentz rotor.

APS uses the same mathematics to find not only passive and active Lorentz rotations, but also any mixture of passive and active rotations: all that counts is the *relative* Lorentz rotation of the observed object with respect to the observer. This property means that the basis paravectors themselves represent not an absolute frame, but rather a frame *relative* to the observer (or Lab). The proper basis $\{\mathbf{e}_\mu\}$ with $\mathbf{e}_0 = 1$ represents a frame at rest with respect to the observer. In APS, as in experiments, it is only the *relative* motion and orientation of the observed object with respect to the observer that is significant.

Covariant Elements and Invariant Properties

Experiments generally measure real scalars such as the size of paravector components on a given basis. The most meaningful geometric quantities in relativity, however, are spacetime vectors and products thereof that simply rotate and reflect in paravector space under the action of Lorentz transformations. Such quantities are said to be *covariant.* In APS, covariant spacetime vectors are real paravectors, and the biparavectors and triparavectors formed from them are also covariant, but one can move back and forth easily between covariant quantities and their components. Individual components are not generally covariant. Some properties, such as the scalar product of covariant paravectors and the square of simple covariant biparavectors, are *invariant,* unchanged by Lorentz rotations. Such properties are known as Lorentz scalars.

Relation of APS to STA

An alternative to the paravector model of spacetime is the spacetime algebra (STA) introduced by David Hestenes[13, 15]. APS and STA are closely related, and it is the purpose of this section show how.

STA is the geometric algebra $C\ell_{1,3}$ of Minkowski spacetime. Whereas the Minkowski spacetime metric appears automatically in APS, it is imposed in STA. In each frame STA starts with a 4-dimensional orthonormal basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \equiv \{\gamma_\mu\}$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu},$$

where as previously, $\eta_{\mu\nu}$ are elements of the metric tensor $(\eta_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, -1)$. The volume element in STA is $\mathbf{I} = \gamma_0\gamma_1\gamma_2\gamma_3$. Although it is referred to as the unit pseudoscalar and squares to -1, it anticommutes with vectors, thus behaving more like an additional spatial dimension than a scalar.

The frame chosen for a system can be independent of the observer and her frame $\{\hat{\gamma}_\mu\}$. Any spacetime vector $p = p^\mu \gamma_\mu$ expanded in the system frame $\{\gamma_\mu\}$ can be multiplied by observer's time axis $\hat{\gamma}_0$ to give

$$p\hat{\gamma}_0 = p \cdot \hat{\gamma}_0 + p \wedge \hat{\gamma}_0 = p^\mu \mathbf{u}_\mu \ , \tag{32}$$

where[3] $\mathbf{u}_\mu = \gamma_\mu \hat{\gamma}_0$. In particular, $\mathbf{u}_0 = \gamma_0 \hat{\gamma}_0$ is the proper velocity of the system frame $\{\gamma_\mu\}$ with respect to the observer frame $\{\hat{\gamma}_\mu\}$. If the system frame is at rest with respect to the observer, then $\gamma_0 = \hat{\gamma}_0$ and

[3] A double arrow might be thought more appropriate than an equality here, because $\mathbf{u}_\mu$ and $\gamma_\mu, \hat{\gamma}_0$ act in different algebras. However, we are identifying $C\ell_3$ with the even subalgebra of $C\ell_{1,3}$, so that the one algebra is embedded in the other. Caution is still needed to avoid statements such as

$$i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \overset{\text{wrong!}}{=} \hat{\gamma}_1 \wedge \hat{\gamma}_0 \wedge \hat{\gamma}_2 \wedge \hat{\gamma}_0 \wedge \hat{\gamma}_3 \wedge \hat{\gamma}_0 = 0 \ .$$

This is not valid because the wedge products on either side of the third equality refer to different algebras and are not equivalent.

the relative basis paravectors $\mathbf{u}_\mu$ are replaced by proper basis elements $\mathbf{e}_\mu$. The result

$$p\hat{\gamma}_0 = p^\mu \mathbf{e}_\mu = p^0 + \mathbf{p},$$

where $\mathbf{e}_0 \equiv 1$ and $\mathbf{e}_k \equiv \gamma_k \hat{\gamma}_0$, is called a *space/time split.* The association $\gamma_\mu \hat{\gamma}_0 = \mathbf{e}_\mu$ when $\gamma_0 = \hat{\gamma}_0$ establishes the previously mentioned isomorphism between the even subalgebra of STA and APS. Together with $\mathbf{u}_\mu = \gamma_\mu \hat{\gamma}_0$ for a more general system basis, it emphasizes that the basis vectors in APS are *relative*: they always relate two frames, the system frame and the observer frame, each of which has its own basis in STA.

Clifford conjugation in APS corresponds to reversion in STA, indicated by a tilde. For example, $\bar{\mathbf{u}}_\mu = (\gamma_\mu \hat{\gamma}_0)^{\tilde{}} = \hat{\gamma}_0 \gamma_\mu$. In particular, the proper velocity of the observer frame $\{\hat{\gamma}_\mu\}$ with respect to γ_0 is $\bar{\mathbf{u}}_0 = \hat{\gamma}_0 \gamma_0$, the inverse of $\mathbf{u}_0 = \gamma_0 \hat{\gamma}_0$. It is not possible to make all of the basis vectors in any STA frame Hermitian, but one usually takes $\hat{\gamma}_0^\dagger = \hat{\gamma}_0$ and $\hat{\gamma}_k^\dagger = -\hat{\gamma}_k$ in the observer's frame $\{\hat{\gamma}_\mu\}$. More generally, Hermitian conjugation of an arbitrary element Γ in STA combines reversion with reflection in the observer's time axis $\hat{\gamma}_0$: $\Gamma^\dagger = \hat{\gamma}_0 \tilde{\Gamma} \hat{\gamma}_0$. For example, the relation

$$\mathbf{u}_\mu^\dagger = \left[\hat{\gamma}_0 \left(\gamma_\mu \hat{\gamma}_0\right)^{\tilde{}}\, \hat{\gamma}_0\right] = \gamma_\mu \hat{\gamma}_0 = \mathbf{u}_\mu$$

shows that all the paravector basis vectors $\mathbf{u}_\mu$ are Hermitian. It is important to note that Hermitian conjugation is frame dependent in STA just as Clifford conjugation of paravectors is in APS.

Example 5 *The Lorentz-invariant scalar part of the paravector product $p\bar{q}$ in APS has the same expansion as in STA:*

$$\begin{aligned}
\langle p\bar{q}\rangle_S &= \frac{1}{2} p^\mu q^\nu \left(\mathbf{e}_\mu \bar{\mathbf{e}}_\nu + \mathbf{e}_\nu \bar{\mathbf{e}}_\mu\right) \\
&= \frac{1}{2} p^\mu q^\nu \left(\gamma_\mu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\nu + \gamma_\nu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\mu\right) \\
&= p^\mu q^\nu \eta_{\mu\nu} .
\end{aligned}$$

Basis biparavectors in APS become basis bivectors in STA:

$$\begin{aligned}
\frac{1}{2}\left(\mathbf{e}_\mu \bar{\mathbf{e}}_\nu - \mathbf{e}_\nu \bar{\mathbf{e}}_\mu\right) &= \frac{1}{2}\left(\gamma_\mu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\nu - \gamma_\nu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\mu\right) \\
&= \frac{1}{2}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right).
\end{aligned}$$

Lorentz transformations in STA are effected by $\gamma_\mu \rightarrow L \gamma_\mu \tilde{L}$, with $L\tilde{L} = 1$. Every product of basis vectors transforms the same way. An *active transformation* keeps the observer frame fixed and transforms only the system frame. Suppose the system frame is related to the observer frame by the Lorentz rotor $L : \gamma_\mu = L\hat{\gamma}_\mu\tilde{L}$. Then

$$\mathbf{u}_\mu = \gamma_\mu\hat{\gamma}_0 = L\hat{\gamma}_\mu\tilde{L}\hat{\gamma}_0 = L\hat{\gamma}_\mu\hat{\gamma}_0\left(\hat{\gamma}_0\tilde{L}\hat{\gamma}_0\right) = L\mathbf{e}_\mu L^\dagger,$$

which coincides with the relation in APS. In a *passive transformation*, it is the system frame that stays the same and the observer's frame that changes. Let us suppose that the observer moves from frame $\{\gamma_\mu\}$ to frame $\{\hat{\gamma}_\mu\}$ where $\gamma_\mu = L\hat{\gamma}_\mu\tilde{L}$. Then

$$\mathbf{e}_\mu = \gamma_\mu\gamma_0 \rightarrow \mathbf{u}_\mu = \gamma_\mu\hat{\gamma}_0 \,.$$

To re-express the transformed relative coordinates $\mathbf{u}_\mu$ in terms of the original $\mathbf{e}_\mu$, we must expand the system frame vectors γ_μ in terms of the observer's transformed basis vectors $\hat{\gamma}_\mu$. Thus

$$\mathbf{u}_\mu = L\hat{\gamma}_\mu\tilde{L}\hat{\gamma}_0 = L\mathbf{e}_\mu L^\dagger.$$

The mathematics is identical to that for the active transformation, but the interpretation is different. It is important to stress that the system and observer frames are distinct in STA, and under active or passive transformations, only one of them changes. Confusion about this point can easily lead to errors in the transformation properties of elements and their relation space/time splits.

Since the transformations can be realized by changing the observer frame and keeping the system frame constant, the physical objects can be taken to be fixed in STA, giving what is sometimes referred to as an *invariant* formulation of relativity. Note, however, that the name "invariant" for covariant objects is consistent only if no active Lorentz transformations are needed. In order to avoid inconsistency and to prevent confusion of covariant expressions with Lorentz scalars such as scalar products of spacetime vectors, I prefer to call STA an *absolute-frame* formulation of relativity.

We have seen that a Lorentz rotation has the same physical effect whether we rotate the object forward or the observer backward or some combination. This is trivially incorporated in APS where only the object frame relative to the observer enters. The absolute frames of STA, while sometimes convenient, impose an added structure not required by experiment.

The space/time split of a property in APS is simply a result of expanding into vector grades in the observer's proper basis $\{\mathbf{e}_\mu\}$:

$$p = p^0 + \mathbf{p}$$
$$\mathbf{F} = \mathbf{E} + i\mathbf{B} \,.$$

Although p and $\mathbf{F}$ are covariant, the split is not; it is valid only in a rest frame of the observer. To relate this to the split as seen in a frame moving with proper velocity u with respect to the observer, we expand p in the paravector basis $\{\mathbf{u}_\mu = L\mathbf{e}_\mu L^\dagger\}$ and $\mathbf{F}$ in the corresponding biparavector basis $\{\langle \mathbf{u}_\mu \bar{\mathbf{u}}_\nu \rangle_V\}$, where $u = \mathbf{u}_0$. The passive transformation of the observer to the moving frame replaces $\mathbf{u}_\mu$ by $\mathbf{e}_\mu$, and the new space/time split simply separates the result into vector grades. The physical fields, momenta, etc. are transformed and are not invariant in APS but *covariant,* that is, under a Lorentz transformation, the form of equations relating them remains the same even though the vectors and multivectors themselves change.[4]

STA uses the metric of signature $(1,3)$, but the pseudoEuclidean metric of signature $(3,1)$ could equally well have been used. This alternative uses the real Clifford algebra $\mathcal{C}\ell_{3,1}$ in place of STA's $\mathcal{C}\ell_{1,3}$.. Because the two algebras are inequivalent, there is no simple transformation between them, and some authors have debated the relative merits of the two choices. APS easily accommodates both possibilities. Our formulation above gives a paravector metric of signature $(1,3)$, but simply by changing the overall sign on the definition of the quadratic form (or, equivalently, of the Clifford conjugate), we obtain a paravector space of the other signature, namely $(3,1)$.

STA and APS seem equally adept at modeling relativistic phenomena. This at first is surprising since STA has $2^4 = 16$ linearly independent elements whereas APS has only half that many. To understand how APS achieves its compactness, note that Lorentz scalars are grade 0 objects in both STA and APS, but spacetime vectors in STA are homogeneous elements of grade 1 whereas in APS they are paravectors, which mix grades 0 and 1. APS maintains a formal grade distinction between an observer's proper time axis and spatial directions by making time a scalar and spatial direction a vector. Furthermore, spacetime planes are represented by elements of grade 2 in STA and by biparavectors, which combine elements of grades 1 and 2, in APS. Elements of a given grade evidently play a double role in APS. Rather than being a disadvantage, however, the double roles mirror common usage.

For example, the spacetime momentum of a particle at rest has only one nonvanishing component, namely a time component equal to its mass ($c = 1$), but the mass is also a Lorentz scalar giving the invariant "length" of the momentum. In APS, the mass is simply a scalar that can fill both roles. In STA, the two roles are represented by expressions

[4] You can have *absolute frames* in APS, if you want them for use in passive transformations, by introducing an *absolute observer*.

of different grades: the Lorentz-invariant mass is a scalar while the rest-frame momentum is a vector. They are not equal but are instead related by a space/time split, which requires multiplication by $\hat{\gamma}_0$.

Another example is provided by the electromagnetic field $\mathbf{F}$, which for a given observer reduces to the electric field $\mathbf{E}$ if there is no magnetic part. In APS, $\mathbf{F}$ is a biparavector with a vector part $\mathbf{E}$ and a bivector part $ic\mathbf{B}$ for any given observer. There is no problem in identifying $\mathbf{E}$ both as a spatial vector and a spacetime plane that includes the time axis. In STA, on the other hand, the two choices require different notation. Since $\mathbf{F}$ is an element of grade 2, we either specify $\mathbf{E}$ as $\mathbf{F}\cdot\hat{\gamma}_0$ or as this times $\hat{\gamma}_0$. The expression $\mathbf{F}\cdot\hat{\gamma}_0$ has the form of a spacetime vector in STA, but of course $\mathbf{E}$ transforms differently. The correct transformation behavior of $\mathbf{F}\cdot\hat{\gamma}_0$ is obtained in STA if, as discussed above, one distinguishes between the observer frame $\{\hat{\gamma}_\mu\}$ and the system frame $\{\gamma_\mu\}$ and applies the Lorentz rotation to only one of them.

The double-role playing of vector grades in APS is responsible for its efficiency in modeling spacetime. STA requires twice as many degrees of freedom to model the same phenomena. This appears to be the cost of having an absolute-frame formulation of relativity. Both STA and APS easily relate a covariant representation to observer-dependent measurements, although the connection is more direct in APS.

4. Eigenspinors

The motion of a particle is described by the special Lorentz rotor $L = \Lambda$ that transforms the particle from rest to the lab. Any property known in the rest frame can be transformed to the lab by Λ, which is known as the *eigenspinor* of the particle and is generally a function of its proper time τ. For example, the spacetime momentum in the lab is $p = \Lambda m \Lambda^\dagger$. The term "spinor" refers to the form of a Lorentz rotation of Λ, namely

$$\Lambda \to L\Lambda, \tag{33}$$

which is the form for the composition of Lorentz rotations but is distinct from Lorentz rotations of paravectors and their products.

The eigenspinor $\Lambda(\tau)$ is the solution of a time evolution equation in the simple linear form

$$\frac{d\Lambda}{d\tau} \equiv \dot{\Lambda} = \frac{1}{2}\mathbf{\Omega}\Lambda(\tau), \tag{34}$$

where $\mathbf{\Omega}$ is the *spacetime rotation rate* of the particle in the lab. This approach offers new tools for classical physics. It implies that

$$\dot{p} = \dot{\Lambda} m \Lambda^\dagger + \Lambda m \dot{\Lambda}^\dagger = \langle \mathbf{\Omega} p \rangle_{\Re} \ . \tag{35}$$

For the motion of a charge e in the electromagnetic field $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, the identification

$$\boldsymbol{\Omega} = \frac{e}{m}\mathbf{F}, \tag{36}$$

when substituted into (35), gives the covariant Lorentz-force equation. Note that (36) can be taken as a covariant definition of $\mathbf{F}$, valid in any inertial frame. It is trivial to find Λ for any uniform field $\mathbf{F}$:

$$\Lambda(\tau) = \exp\left(\frac{e\mathbf{F}\tau}{2m}\right)\Lambda(0). \tag{37}$$

Solutions can also be found for relativistic charge motion in plane waves, plane-wave pulses, or plane waves superimposed on static longitudinal electric or magnetic fields.[19]

5. Maxwell's Equation

Maxwell's famous equations are inherently relativistic, and it is a shame that so many texts treat much of electrodynamics in nonrelativistic approximation. One reason given for introducing relativity only late in an electrodynamics course is that tensors and or matrices are required which makes the presentation more abstract and harder to interpret. In APS we can easily display and exploit relativistic symmetries in simple vector and paravector terms without the need of tensors or matrices.

Maxwell's equations were written as a single quaternionic equation by Conway (1911), Silberstein (1912), and others. In APS we can write

$$\bar{\partial}\mathbf{F} = \mu_0 \bar{j}\ , \tag{38}$$

where $\mu_0 = \varepsilon_0^{-1} = 4\pi \times \dot{3}0$ Ohm is the impedance of the vacuum, with $\dot{3} \equiv 2.99792458$. The usual four equations are simply the four vector grades of this relation, extracted as the real and imaginary, scalarlike and vectorlike parts. It is also seen as the *necessary covariant extension* of Coulomb's law $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$. The covariant field is not $\mathbf{E}$ but $\mathbf{F} = \mathbf{E}+i\mathbf{B}$, the divergence is part of the covariant gradient $\bar{\partial}$, and ρ must be part of $\bar{j} = \rho - \mathbf{j}$. The combination is Maxwell's equation.[5]

It is a simple exercise to derive the continuity equation $\langle \partial\bar{j}\rangle_S = 0$ in one step from Maxwell's equation. One need only note that the D'Alembertian $\partial\bar{\partial}$ is a scalar operator and that $\langle \mathbf{F}\rangle_S = 0$.

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[5] We have assumed that the source is a real paravector current and that there are no contributing pseudoparavector currents. Known currents are of the real paravector type, and a pseudoparavector current would behave counter-intuitively under parity inversion. Our assumption is supported experimentally by the apparent lack of magnetic monopoles.

Directed Plane Waves

In source-free space ($\bar{j} = 0$), there are solutions $\mathbf{F}(s)$ that depend on spacetime position only through the Lorentz invariant $s = \langle k\bar{x}\rangle_S = \omega t - \mathbf{k}\cdot\mathbf{x}$, where $k = \omega + \mathbf{k} \neq 0$ is a constant propagation paravector. Since $\partial \langle k\bar{x}\rangle_S = k$, Maxwell's equation gives

$$\bar{\partial}\mathbf{F} = \bar{k}\mathbf{F}'(s) = 0 \ . \tag{39}$$

In a division algebra, we could divide by $\bar{k}$ and conclude that $\mathbf{F}'(s) = 0$, a rather uninteresting solution. There is another possibility here because APS is not a division algebra: $\bar{k}$ may have no inverse. Then k has the form $k = \omega\left(1 + \hat{\mathbf{k}}\right)$, and after integrating (39) from some s_0 at which $\mathbf{F}$ is presumed to vanish, we get $\left(1 - \hat{\mathbf{k}}\right)\mathbf{F}(s) = 0$, which means $\mathbf{F}(s) = \hat{\mathbf{k}}\mathbf{F}(s)$. The scalar part of $\mathbf{F}$ vanishes and consequently $\left\langle \hat{\mathbf{k}}\mathbf{F}(s)\right\rangle_S = \hat{\mathbf{k}}\cdot\mathbf{F}(s) = 0$ so that the fields $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to $\hat{\mathbf{k}}$ and thus anticommute with it. Furthermore, equating imaginary parts gives $i\mathbf{B} = \hat{\mathbf{k}}\mathbf{E}$ and it follows that

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \left(1 + \hat{\mathbf{k}}\right)\mathbf{E}(s) \tag{40}$$

with $\mathbf{E} = \langle \mathbf{F}\rangle_{\Re}$ real. This is a plane-wave solution with $\mathbf{F}$ constant on spatial planes perpendicular to $\hat{\mathbf{k}}$. Such planes propagate at the speed of light along $\hat{\mathbf{k}}$. In spacetime, $\mathbf{F}$ is constant on the light cone $\hat{\mathbf{k}}\cdot\mathbf{x} = t$. However, $\mathbf{F}$ is not necessarily monochromatic, since $\mathbf{E}(s)$ can have any functional form, including a pulse, and the scale factor ω, although it has dimensions of frequency, may have nothing to do with any physical oscillation. The structure of the plane wave $\mathbf{F}$ is that of a simple biparavector representing the spacetime plane containing both the null paravector $1 + \hat{\mathbf{k}}$ and the orthogonal direction $\mathbf{E}$. This structure ensures that $\mathbf{F}$ itself is *null*: $\mathbf{F}^2 = 0$. In fact, $\mathbf{F}$ is what Penrose calls a *null flag*. The *flagpole* $1 + \hat{\mathbf{k}}$ lies on the light cone in the plane of the flag but is orthogonal to both itself and the flag. This is the basis of an important symmetry that is critical for determining charge dynamics in plane waves.[19] The null-flag structure is beautiful and powerful, but you miss it entirely if only write out separate electric and magnetic fields. The electric and magnetic fields are simply components of the null flag; $\mathbf{E}$ gives the extent of the flag perpendicular to the flagpole, and $\mathbf{B}$ represents the spatial plane swept out by $\mathbf{E}$ as it propagates along $\hat{\mathbf{k}}$.

The electric field $\mathbf{E}(s)$ determines the polarization of the wave. If the direction of $\mathbf{E}$ is constant, for example $\mathbf{E}(s) = \mathbf{E}(0)\cos s$, the wave is linearly polarized along $\mathbf{E}(0)$. If $\mathbf{E}$ rotates around $\hat{\mathbf{k}}$, for example

$\mathbf{E}(s) = \mathbf{E}(0)\exp\left(i\kappa\hat{\mathbf{k}}s\right)$, $\kappa = \pm 1$, the wave is circularly polarized with helicity κ. Note that the flagpole can gobble the unit vector $\hat{\mathbf{k}}$:

$$\mathbf{F}(s) = \left(1+\hat{\mathbf{k}}\right)\mathbf{E}(0)\exp\left(i\kappa\hat{\mathbf{k}}s\right) = \left(1+\hat{\mathbf{k}}\right)\exp\left(-i\kappa\hat{\mathbf{k}}s\right)\mathbf{E}(0)$$
$$= \left(1+\hat{\mathbf{k}}\right)\mathbf{E}(0)\exp\left(-i\kappa s\right). \tag{41}$$

This establishes an equivalence for null flags between rotations about the spatial direction $\hat{\mathbf{k}}$ of the flagpole and multiplication by a phase factor. Since the dual of $\mathbf{F}$ is $-i\mathbf{F}$, the phase factor is said to induce duality rotations. The energy density $\mathcal{E} = \frac{1}{2}\left(\varepsilon_0\mathbf{E}^2 + \mathbf{B}^2/\mu_0\right)$ and the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ for the plane wave are combined in

$$\frac{1}{2}\varepsilon_0\mathbf{F}\mathbf{F}^\dagger = \mathcal{E} + \mathbf{S} = \varepsilon_0\mathbf{E}^2\left(1+\hat{\mathbf{k}}\right).$$

Polarization Basis

The application of APS to the polarization of a beam of light or other electromagnetic radiation gives a formulation vastly simpler than the usual approach with Mueller matrices, and it demonstrates spinorial type transformations and additional uses of paravectors. Furthermore, the mathematics is the same as used to describe electron polarization, which I discuss in my next lecture.

The field $\mathbf{F}$ of a beam of monochromatic radiation can be expressed as a linear combination of two independent polarization types. Both linear and circular polarization bases are common, but a circular basis is most convenient, partially because of the relation noted above between spatial and duality rotations. Circularly polarized waves also have the simple form used popularly by R. P. Feynman[20] to discuss light propagation in terms of a rotating pointer that we can take to be $\mathbf{E}(s)$. A linear combination of both helicities of a directed plane wave gives

$$\mathbf{F} = \left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0 e^{i\delta\hat{\mathbf{k}}}\left(E_+ e^{is\hat{\mathbf{k}}} + E_- e^{-is\hat{\mathbf{k}}}\right)$$
$$= \left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0 e^{-i\delta}\left(E_+ e^{-is} + E_- e^{is}\right),$$

where $E_\pm$ are the real field amplitudes, δ gives the rotation of $\mathbf{E}$ about $\hat{\mathbf{k}}$ at $s = 0$from the unit vector $\hat{\mathbf{E}}_0$, and in the second line, we gobbled $\hat{\mathbf{k}}$'s. Because every directed plane wave can be expressed in the form $\mathbf{F} = \left(1+\hat{\mathbf{k}}\right)\mathbf{E}(s)$, it is sufficient to determine the electric field $\mathbf{E}(s) =$

$\langle \mathbf{F} \rangle_{\Re}$:

$$\begin{aligned}\mathbf{E} &= \left\langle \left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0 E_+ e^{-i\delta} e^{-is} + \left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0 E_- e^{-i\delta} e^{is} \right\rangle_{\Re} \\ &= \left\langle \left[\left(1+\hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0 E_+ e^{-i\delta} + \hat{\mathbf{E}}_0 \left(1+\hat{\mathbf{k}}\right) E_- e^{i\delta}\right] e^{-is} \right\rangle_{\Re} \\ &= \left\langle \left(\boldsymbol{\epsilon}_+, \boldsymbol{\epsilon}_-\right) \Phi e^{-is} \right\rangle_{\Re} ,\end{aligned}$$

where the complex polarization basis vectors $\boldsymbol{\epsilon}_{\pm} = 2^{-1/2}\left(1 \pm \hat{\mathbf{k}}\right)\hat{\mathbf{E}}_0$ are null flags satisfying $\boldsymbol{\epsilon}_- = \boldsymbol{\epsilon}_+^{\dagger},\ \boldsymbol{\epsilon}_+ \cdot \boldsymbol{\epsilon}_+^{\dagger} = 1 = \boldsymbol{\epsilon}_- \cdot \boldsymbol{\epsilon}_-^{\dagger}$, and the *Poincaré spinor*

$$\Phi = \sqrt{2}\begin{pmatrix} E_+ e^{-i\delta} \\ E_- e^{i\delta} \end{pmatrix} \tag{42}$$

gives the (real) electric-field amplitudes and their phases, and it contains all the information needed to determine the polarization and intensity of the wave. The spinor (42) is related by unitary transformation to the Jones vector, which uses a basis of linear polarization.

Stokes Parameters. To describe partially polarized light, we can use the *coherency density,*[6] which in the case of a single Poincaré spinor is

$$\rho = \varepsilon_0 \Phi \Phi^{\dagger} = \rho^{\mu} \boldsymbol{\sigma}_{\mu} , \tag{43}$$

where the $\boldsymbol{\sigma}_{\mu}$ are the Pauli spin matrices. The normalization factor ε_0 has been chosen to make ρ^0 the time-averaged energy density. The coefficients $\rho^{\mu} = \langle \rho \boldsymbol{\sigma}_{\mu} \rangle_S$ are the *Stokes parameters.* The coherency density $\rho = \rho^0 + \boldsymbol{\rho}$ is a paravector in the algebra for the space spanned by $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ .This space, called *Stokes space,* is a 3-D Euclidean space analogous to physical space. It is not physical space, but its geometric algebra is isomorphic to APS, and it illustrates how Clifford algebras can arise in physics for spaces other than physical space.

As defined for a single Φ, ρ is null ($\det \rho = \rho \bar{\rho} = 0$) and can be written $\rho = \rho^0 (1 + \mathbf{n})$,where $\mathbf{n}$, a unit vector in the direction of $\boldsymbol{\rho}$, specifies the type of polarization. In particular, for positive helicity light, $\boldsymbol{n} = \boldsymbol{\sigma}_3$, for negative helicity polarization $\boldsymbol{n} = -\boldsymbol{\sigma}_3$, and for linear polarization at an angle $\delta = \phi/2$ with respect to $\mathbf{E}_0$, $\boldsymbol{n} = \boldsymbol{\sigma}_1 \cos\phi + \boldsymbol{\sigma}_2 \sin\phi$. Other directions correspond to elliptical polarization.

Polarizers and Phase Shifters. The action of ideal polarizers and phase shifters on the wave is modeled mathematically by transformations on the Poincaré spinor Φ of the form $\Phi \to T\Phi$. For *polarizers* T that pass polarization of type $\mathbf{n}$, T is the *projector* $T = \mathsf{P}_{\mathbf{n}} = \frac{1}{2}(1 + \mathbf{n})$,which is proportional to the pure state of that polarization. Projectors are idempotent ($\mathrm{P}_{\mathbf{n}}^2 = \mathrm{P}_{\mathbf{n}}$), just as we would expect for ideal polarizers since a second application of $\mathsf{P}_{\mathbf{n}}$ changes nothing further. The polarizer represented by the complementary projector $\bar{\mathsf{P}}_{\mathbf{n}}$ annihilates $\mathrm{P}_{\mathbf{n}}$: $\bar{\mathsf{P}}_{\mathbf{n}}\mathsf{P}_{\mathbf{n}} = \mathsf{P}_{-\mathbf{n}}\mathsf{P}_{\mathbf{n}} = 0$, and in general, opposite directions in Stokes space correspond to orthogonal polarizations.

If the wave is split into orthogonal polarization components ($\pm\mathbf{n}$) and the two components are given a relative phase shift of α, the result is equivalent to rotating $\boldsymbol{\rho}$ by α about $\mathbf{n}$ in Stokes subspace: $T = \mathsf{P}_{\mathbf{n}}e^{i\alpha/2} + \bar{\mathsf{P}}_{\mathbf{n}}e^{-i\alpha/2} = e^{i\mathbf{n}\alpha/2}$. Depending on $\mathbf{n}$, this operator can represent both the Faraday effect and the effect of a birefringent medium with polarization types $\mathbf{n}$ and $-\mathbf{n}$ corresponding to the slow and fast axes, respectively.

Coherent Superpositions and Incoherent Mixtures. A superposition of two waves of the same frequency is *coherent* because their relative phase is fixed. Mathematically, one adds spinors in such cases: $\Phi = \Phi_1 + \Phi_2$, where the subscripts refer to the two waves, not to spinor components. Waves of different frequencies have a continually changing relative phase, and when averaged over periods large relative to their beat period, combine *incoherently*: $\rho = \rho_1 + \rho_2$. The degree of polarization is given by the length of $\boldsymbol{\rho}$ relative to ρ_0 and can vary from 0 to 100%. Any transformation T of spinors, $\Phi \to T\Phi$, transforms the coherency density by $\rho \to T\rho T^\dagger$, and transformations that do not preserve the polarization can also be applied to ρ.[6]

6. Conclusions

The multiparavector structure of APS makes the algebra ideal for modeling relativistic phenomena. It presents a covariant formulation based on relative motion and orientation but provides a simple path to the spatial vectors for any given observer as well as to the operators that act on the vectors. The formulation is simple enough to be used in introductory physics courses, and it holds the promise of becoming a key factor in any curriculum revision designed to train students to contribute significantly to the quantum age of the 21st century.

Since APS is isomorphic to complex quaternions, any calculation in APS can be repeated with quaternions taken over the complex field. However, the geometry is considerably clearer in APS. In its descriptive and computational power for relativistic physics, APS seems as capable as STA. However, STA has twice the size of APS in order to add the non-observable structure of absolute frames to its formulation.

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