QUANTUM/CLASSICAL INTERFACE: A GEOMETRIC APPROACH FROM THE CLASSICAL SIDE

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Abstract Classical relativistic physics in Clifford's geometric algebra has a spinorial formulation that is closely related to the standard quantum formalism. The algebraic use of spinors and projectors, together with the bilinear relations of spinors to observed currents, gives quantum-mechanical form to many classical results, and the clear geometric content of the algebra makes it an illuminating probe of the quantum/classical interface. The aim of this lecture is to close the conceptual gap between quantum and classical phenomena while highlighting their essential differences. The paravector representation of spacetime in APS is used in particular to provide insight into what many pioneering quantum physicists have considered classically indescribable: spin-1/2 systems and their measurement.

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1. Introduction

This lecture continues the theme[1] that Clifford's geometric algebras, in particular the algebra of physical space (APS), provide the lubricant for smooth paradigm shifts from Newtonian to post-Newtonian paradigms. The focus here is on the shift to the quantum view of the universe, and how APS can make the quantum/classical interface more transparent.

After a brief review of paravector space and its ability to model spacetime, we start the lecture by relating the classical eigenspinor of a particle, that is the Lorentz rotor that takes it from rest to its motion in the

lab, to its relativistic wave function. We then show how the 2-valued measurement of the spin arises naturally from the algebraic properties of rotors, and how the g factor and magnetic moment of elementary fermions result. The spin itself arises classically as a gauge freedom of the eigenspinor, and as we will see, the choice of a spin rate proportional to the total energy of the system leads to de Broglie waves and momentum eigenstates of the Dirac equation. Finally, we consider work on entangled spin states which are of interest in quantum computation.

2. Paravector Space as Spacetime

Recall from my previous lecture[1] that we use paravector space in APS to model spacetime. In addition to the three dimensions of physical space, paravector space also includes scalars. A typical paravector can be written

$$p = p^0 + \mathbf{p} \equiv p^\mu \mathbf{e}_\mu \ , \tag{1}$$

where p^0 is a scalar and $\mathbf{p}$ a physical vector, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of physical space, and $\mathbf{e}_0 = 1$. The *paravector basis* $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the observer's *proper* basis (at rest). Paravector space in APS is thus a four-dimensional linear space. It has a metric determined by the quadratic form, which for the paravector p is proportional to the scalar $p\bar{p} = \bar{p}p = \left(p^0\right)^2 - \mathbf{p}^2$, where $\bar{p} = p^0 - \mathbf{p}$ is the *Clifford conjugate* of p. Since the quadratic form is a scalar expression, insertion of the basis expansion (1) gives

$$p\bar{p} = \langle p\bar{p} \rangle_S = p^\mu p^\nu \left\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \right\rangle_S \equiv p^\mu p^\nu \eta_{\mu\nu}, \tag{2}$$

and the metric tensor $\eta_{\mu\nu} = \left\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \right\rangle_S$ automatically arising here is the *Minkowski metric* of spacetime.

Rotations in paravector space preserve the "square length" $p\bar{p}$ of paravectors and therefore also the scalar products $\langle p\bar{q} \rangle_S = \frac{1}{2}\left(p\bar{q} + q\bar{p}\right)$ between paravectors. They have the form often called a *spin transformation*[2]

$$p \rightarrow LpL^\dagger, \tag{3}$$

where L is a unimodular element ($L\bar{L} = 1$) known as a *Lorentz rotor.* Lorentz rotations are the physical Lorentz transformations of relativity: boosts, spatial rotations, and their products. Several Lorentz rotations can be applied in succession, for example $p \rightarrow L_3 L_2 L_1 p L_1^\dagger L_2^\dagger L_3^\dagger$. All the information about the transformation is contained in the composition $L = L_3 L_2 L_1$,which of course is another Lorentz rotor. Indeed, the Lorentz rotors form a group isomorphic to $SL\left(2, \mathbb{C}\right)$,the double covering group of $SO_+\left(1, 3\right)$.

Lorentz rotors L are generated by biparavectors:

$$L = \pm \exp\left(\mathbf{W}/2\right) \in \$pin_+\left(1,3\right) \simeq SL\left(2,\mathbb{C}\right),$$

where $\mathbf{W}$ is a biparavector

$$\mathbf{W} = \frac{1}{2} W^{\mu\nu} \left\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \right\rangle_V \ .$$

If $\mathbf{W}$ is a vector (a timelike biparavector), then $L = B = B^\dagger$ is a boost, whereas if $\mathbf{W}$ is a bivector (a spacelike biparavector), then $L = R = \bar{R}^\dagger$ is a spatial rotation. Every rotor L can be uniquely factored into the product of a boost and a rotation[3]:

$$L = BR \ .$$

The spin transformations of paravector products are found by transforming each paravector factor (3).

3. Eigenspinors

Recall[1, 4] that the *eigenspinor* Λ of a system is an *amplitude* for the motion and orientation of the system. More precisely, it is a Lorentz rotor of particular interest, relating the system reference frame to the observer (or lab). It transforms distinctly from paravectors and their products: a Lorentz transformation L applied in the lab transforms $\Lambda \to L\Lambda$. This characterizes Λ as a *spinor*. The *eigen* part of its name refers to its association with the system, which in the simplest case is a single particle. The eigenspinor is generally a reducible element of the spinor carrier space of Lorentz rotations. Any property of the system in the reference frame is easily transformed by Λ to the lab. For example, in units with $c = 1$ the proper velocity of a massive particle is taken to be $u = \mathbf{e}_0 = 1$ in the reference frame; that is, the reference frame at proper time τ is an inertial frame that moves with the particle at that instant. In the lab, the proper velocity is

$$u = \Lambda \mathbf{e}_0 \Lambda^\dagger = \Lambda\Lambda^\dagger, \tag{4}$$

which is also the timelike basis paravector of a frame moving with proper velocity u . The complete paravector basis of the moving frame (relative to the observer) is $\left\{\mathbf{u}_\mu = \Lambda \mathbf{e}_\mu \Lambda^\dagger\right\}$ with $u \equiv \mathbf{u}_0$. Thought of as a passive transformation, Λ transforms the observer from the reference frame to the lab.

Time evolution

The eigenspinor Λ of an accelerating or rotating system changes in time, and we let $\Lambda\left(\tau\right)$ be the Lorentz rotation at proper time τ from the

reference frame to the commoving inertial frame of the system. Eigenspinors at different times are related by

$$\Lambda\left(\tau_2\right) = L\left(\tau_2, \tau_1\right)\Lambda\left(\tau_1\right),$$

where by the group property of rotations, the time-evolution operator

$$L\left(\tau_2, \tau_1\right) = \Lambda\left(\tau_2\right)\bar{\Lambda}\left(\tau_1\right)$$

is another Lorentz rotation. The time evolution of Λ is found by solving an equation of motion

$$\dot{\Lambda} = \frac{1}{2}\mathbf{\Omega}\Lambda = \frac{1}{2}\Lambda\mathbf{\Omega}_{\text{ref}}\ , \tag{5}$$

where the unimodularity of Λ ($\Lambda\bar{\Lambda} = 1$) implies that $\mathbf{\Omega} \equiv 2\dot{\Lambda}\bar{\Lambda} = -2\Lambda\overline{\dot{\Lambda}}$ is a biparavector that can be identified as the instantaneous *spacetime rotation rate* of the system as seen in the lab, and $\mathbf{\Omega}_{\text{ref}} = \bar{\Lambda}\mathbf{\Omega}\Lambda$ is its value in the reference frame. Note generally that operators from the left on Λ act in the lab whereas those from the right act in the reference frame.

The equation of motion (5) allows us to compute time-rates of change of any covariant property known in the reference frame in terms of $\mathbf{\Omega}$ or $\mathbf{\Omega}_{\text{ref}}$. For example, the acceleration in the lab can be expanded $\dot{u} = d\left(\Lambda\Lambda^\dagger\right)/d\tau = \dot{\Lambda}\Lambda^\dagger + \Lambda\dot{\Lambda}^\dagger = 2\left\langle\dot{\Lambda}\Lambda^\dagger\right\rangle_{\Re}$ and substitution of the equation of motion (5) gives

$$\dot{u} = \left\langle\mathbf{\Omega}u\right\rangle_{\Re}, \tag{6}$$

Here, $\left\langle x\right\rangle_{\Re} = \frac{1}{2}\left(x + x^\dagger\right)$ indicates the *real* (grades $0 + 1$) part of x. In terms of $\mathbf{\Omega}_{\text{ref}}$,

$$\dot{u} = \left\langle\Lambda\mathbf{\Omega}_{\text{ref}}\Lambda^\dagger\right\rangle_{\Re} = \Lambda\left\langle\mathbf{\Omega}_{\text{ref}}\right\rangle_{\Re}\Lambda^\dagger. \tag{7}$$

Note that relation (7) has the form of a Lorentz rotation of a paravector from the reference frame to the lab. Evidently the real (vector) part of the rotation rate $\mathbf{\Omega}_{\text{ref}}$ in the reference frame is just the acceleration of the system there. Equation (6) is just the covariant Lorentz-force equation if we identify $\mathbf{\Omega} = e\mathbf{F}/m$. This identification provides a convenient covariant operational definition of the electromagnetic field $\mathbf{F}$. However, this definition is more restrictive than is *required* by the Lorentz-force equation, and as we see below, a generalization is needed in the presence of spin.

The equation of motion (5) for eigenspinors, especially when supplemented by algebraic projectors, offers powerful new tools for finding classical trajectories of charges in electromagnetic fields. Advantages of the eigenspinor approach are nicely demonstrated in new analytic solutions for the autoresonant laser accelerator.[5]

Gauge Transformations

The eigenspinor equation (5) embodies more information than can ever be obtained from the Lorentz force equation alone: Λ gives not only the velocity $\Lambda\Lambda^\dagger$ of the particle, but also its *orientation*. In this section we look at the gauge freedom associated with orientation and see that it leads to the concept of spin.

The proper velocity $u = \Lambda\Lambda^\dagger$ of a massive particle is invariant under transformations of the form

$$\Lambda \rightarrow \Lambda R_0 \ , \tag{8}$$

where R_0 is any spatial rotation: $R_0 R_0^\dagger = R_0 \bar{R}_0 = 1$. Because R_0 is positioned on the right of Λ, it acts on the reference frame. Transformations that leave measured values unchanged are called *gauge transformations,* and if, as in Newtonian mechanics, we measure only the spacetime position (world line) of the particle and derivatives thereof, then the reorientation (8) of the reference frame is a gauge transformation. We therefore have the gauge freedom to choose the reference-frame orientation as we wish.

The rotation R_0 in (8) may be time dependent. Then, in order for the equation of motion (5) to remain valid, not only is the spacetime rotation rate in the reference frame transformed to the new reference frame, but it also adds an imaginary (bivector) part:

$$\boldsymbol{\Omega}_{\text{ref}} \rightarrow \bar{R}_0 \left(\boldsymbol{\Omega}_{\text{ref}} - i\boldsymbol{\omega}_0\right) R_0 \ , \tag{9}$$

where $-i\boldsymbol{\omega}_0 = 2\dot{R}_0\bar{R}_0$ is the spatial rotation rate that generates R_0. We can confirm from Eq. (7) that the two transformations (8) and (9) together leave the acceleration (and by integration, the proper velocity and the world line) of the massive particle unchanged.

Free De Broglie Waves

The gauge freedom of the classical eigenspinor equations thus admits an arbitrary rotation of the reference frame. It is reasonable to assume that for a free particle the rotation rate $\boldsymbol{\omega}_0$ is constant, and since we are free to choose the initial orientation of the reference frame, we can take $\boldsymbol{\omega}_0 = \omega_0 \mathbf{e}_3$. This rotation rate generates what we refer to as a "spin" rotation, with the spin rotor $R_0 = \exp\left(-i\mathbf{e}_3\omega_0\tau/2\right)$ acting in the reference frame. The eigenspinor of the free particle then has the form

$$\Lambda\left(\tau\right) = \Lambda\left(0\right) e^{-i\mathbf{e}_3\omega_0\tau/2} \tag{10}$$

and satisfies the equation of motion

$$\dot{\Lambda} = \frac{1}{2}\Lambda\boldsymbol{\Omega}_{\text{ref}} \ , \ \boldsymbol{\Omega}_{\text{ref}} = -i\omega_0\mathbf{e}_3.$$

The linear form of the equation of motion (5) generally suggests the use of superpositions of solutions with different world lines, and hence with different proper times τ. To be able to combine these we need to express the proper time τ in particle coordinates x :

$$\tau = \langle x\bar{u}\rangle_S \ \ = \langle u\bar{x}\rangle_S \, , \tag{11}$$

where u is the proper velocity of the particle. When τ (11) is substituted in (10), the result has the form of *free de Broglie waves* in covariant formulation, except that the phase is now a real angle of the spin rotation. The wavelength is fixed at the de Broglie value by relating the rest energy m of the particle to ω_0 by

$$m = \frac{1}{2}\hbar\omega_0 \ .$$

In terms of the momentum paravector $p = mu$, the free eigenspinors (10) are then

$$\Lambda\left(x\right) = \Lambda\left(0\right)\exp\left[-i\mathbf{e}_3\left\langle p\bar{x}\right\rangle_S/\hbar\right]. \tag{12}$$

4. Spin

The experimentally verified existence of de Broglie waves suggests that the spin rotation in the free eigenspinor (12) is real and occurs at the high angular frequency associated with Zitterbewegung:

$$\omega_0 = 2m/\hbar = 1.55\times10^{21}\,\mathrm{s}^{-1} \, .$$

As introduced above, the proper paravector basis $\{\mathbf{e}_\mu\}$ of the particle is transformed to the basis $\left\{\mathbf{u}_\mu = \Lambda\mathbf{e}_\mu\Lambda^\dagger\right\}$ of a *commoving particle frame* (relative to the observer) by the Lorentz transformation Λ. In particular the moving time axis $\mathbf{u}_0 = u = \Lambda\Lambda^\dagger$ is the proper velocity of that frame with respect to the observer. It is natural to ask whether any of the transformed *spatial* vectors, $\mathbf{u}_k = \Lambda\mathbf{e}_k\Lambda^\dagger$, carries any special physical meaning. By taking $\mathbf{e}_3$ to be the rotation axis in the reference frame, the corresponding unit paravector $\mathbf{u}_3$ is the only spatial axis of the commoving frame that is not spun into a blur in the lab. It is associated with the *spin.* We refer to the unit paravector $\mathbf{u}_3 = w$ as the (unit) *spin paravector*[1] and note that $w\bar{w} = -1$.

[1] The physical spin requires an extra scalar factor of $\hbar/2$ to ensure the correct magnitude and units.

Since $\mathbf{e}_0$ and $\mathbf{e}_3$ are orthogonal, so are $\mathbf{u}_0$ and $\mathbf{u}_3$ and hence u and w :

$$\langle \mathbf{u}_3 \bar{\mathbf{u}}_0 \rangle_S = \langle w\bar{u} \rangle_S = \langle \mathbf{e}_3 \bar{\mathbf{e}}_0 \rangle_S = 0,$$

and $\mathbf{u}_3 \bar{\mathbf{u}}_0 = w\bar{u}$ is a biparavector (a spacetime plane). Its dual is another biparavector

$$\begin{aligned} \mathbf{S} &= \Lambda \mathbf{e}_1 \bar{\mathbf{e}}_2 \bar{\Lambda} = -i\Lambda \mathbf{e}_3 \bar{\Lambda} \\ &= -iw\bar{u} \ , \end{aligned} \tag{13}$$

which in a frame commoving with the charge is the spin plane. The dual of the spin paravector $w = i\mathbf{S}u = w^\dagger$ is seen to be the *triparavector* $\mathbf{S}u$. This identifies w as the *Pauli-Lubański spin* paravector.

Factoring Λ

Before considering the equation of motion for the spin, recall that as with any Lorentz rotor, Λ can always be factored into the product of a real boost $B = B^\dagger$ and a unitary spatial rotation $R = \bar{R}^\dagger : \Lambda = BR$. The proper velocity of the particle is $u = \Lambda\Lambda^\dagger = B^2$. It follows that the boost part of the eigenspinor is

$$B = u^{1/2} = \frac{1+u}{\sqrt{2\langle 1+u \rangle_S}} = \frac{m+p}{\sqrt{2m\,(m+E)}}, \tag{14}$$

where $p = E + \mathbf{p} = mu$.

The second equality in (14) can be derived simply from the relations $u^2 = B,\ B\bar{B} = 1$ and $B + \bar{B} = \langle 2B \rangle_S$, since then $B\left(B + \bar{B}\right) = B\langle 2B \rangle_S = u + 1$, the scalar part of which determines $\langle B \rangle_S^2$. However, there is a more general geometrical derivation makes the result easy to comprehend and remember. Recall from my previous lecture that any simple Lorentz rotation transforms a paravector v as

$$v \to u = LvL^\dagger = L^2 v^\triangle + v^\perp,$$

where $v^\triangle$ is the part of v coplanar with the paravector rotation plane of L. If this plane contains v, $v^\triangle = v$ and $u = L^2 v$. Let v be a timelike paravector of unit "length": $v\bar{v} = 1$. It is the time axis of a frame moving at proper velocity v (relative to the lab). Then Lv is the unit paravector in the plane of v and u, halfway between them; it is the result of rotating v with only half of the rotation parameter needed to get it to u.Thus, Lv should be parallel to $v + u$. Since it should have the same unit "length" as v, it must be

$$Lv = \frac{v+u}{\sqrt{(v+u)\,(\bar{v}+\bar{u})}} = \frac{v+u}{\sqrt{2\,(1+\langle u\bar{v} \rangle_S)}}$$

and therefore

$$L = (u\bar{v})^{1/2} = \frac{1 + u\bar{v}}{\sqrt{2\left(1 + \langle u\bar{v}\rangle_S\right)}}.$$

In the special case above (14), $v = 1$.

Spin Precession and the *g*-Factor

The charge-to-mass ratio of a classical charged particle is independent of its magnetic moment. The former determines the cyclotron frequency of the charge in a magnetic field, and the latter determines its precession rate. The magnetic moment of a spinning charge is proportional to its angular momentum, and the proportionality constant is given by the g-factor times the charge-to-mass ratio. Consequently, the g-factor determines the ratio of the precession rate to cyclotron frequency of a spin-1/2 fermion in a magnetic field.

The eigenspinor of a particle describes both its translational and rotational motion. Compound systems with several particles or other independent parts may require several eigenspinors to describe their motion. but a classical "elementary" particle can be defined as one whose motion is fully described by a *single* eigenspinor. As we show below, the equation of motion (5) for the eigenspinor remains linear for this eigenspinor only if $g = 2$.

The reference frame of a massive spinning particle at proper time τ, related to the lab by the eigenspinor $\Lambda(\tau)$, is taken to be a commoving inertial frame of the particle. It is a rest frame of the particle in that the proper velocity of the particle in its reference frame is 1. We assume that the eigenspinor Λ includes the spin rotor $R_0 = \exp(-i\mathbf{e}_3\omega_0\tau/2)$. Because of its spin rotation, its reference frames at slightly different τ are rotated by large angles with respect to one another. For many purposes, it is convenient to consider a *nonspinning rest frame,* which is intermediate between the reference and lab frames. Such a frame is defined to be related to the lab frame by a pure boost B. The eigenspinor Λ is the product BR of this boost with the spatial rotation R that rotates the reference frame into the nonspinning rest frame. Under most circumstances, R is dominated by the spin rotor R_0.

The equation of motion (5) can now be written in a third way

$$\dot{\Lambda} = \frac{1}{2}\mathbf{\Omega}\Lambda = \frac{1}{2}\Lambda\mathbf{\Omega}_{\rm ref} = \frac{1}{2}B\mathbf{\Omega}_{\rm rest}R, \tag{15}$$

where the spacetime rotation rate $\mathbf{\Omega}_{\rm rest}$ in the nonspinning rest frame is related to that in the lab ($\mathbf{\Omega}$) and in the reference frame ($\mathbf{\Omega}_{\rm ref}$) by

$$\mathbf{\Omega}_{\rm rest} = \bar{B}\mathbf{\Omega}B = R\mathbf{\Omega}_{\rm ref}\bar{R}\ . \tag{16}$$

The resulting proper acceleration (7)

$$\dot{u} = \langle \mathbf{\Omega} u \rangle_{\Re} = \Lambda \langle \mathbf{\Omega}_{\rm ref} \rangle_{\Re} \Lambda^\dagger = B \langle \mathbf{\Omega}_{\rm rest} \rangle_{\Re} B \tag{17}$$

is consistent with the Lorentz-force equation if and only if

$$\langle \mathbf{\Omega}_{\rm rest} \rangle_{\Re} = e \langle \mathbf{F}_{\rm rest} \rangle_{\Re} / m, \tag{18}$$

where $\mathbf{E}_{\rm rest} = \langle \mathbf{F}_{\rm rest} \rangle_{\Re}$ is the electric field in the nonspinning rest frame. The minimal covariant extension of (18) is $\mathbf{\Omega} = e\mathbf{F}/m$, but this is not acceptable in the presence of spin since the spin rotation rate remains even when the electromagnetic field $\mathbf{F}$ vanishes. Indeed, according to (9), because of the spin term, $\mathbf{\Omega}_{\rm rest}$ contains an extra bivector (imaginary) term $-i\omega_0 R\mathbf{e}_3 R^\dagger = -i\omega_0 \mathbf{s}$, where

$$\mathbf{s} = R\mathbf{e}_3 R^\dagger \tag{19}$$

is the unit spin paravector in the nonspinnng rest frame. We therefore need to consider other covariant extensions of the relation (18).

The Lorentz-force equation is independent of the magnetic field in the rest frame of the charge. We need another equation of motion to relate $\mathbf{\Omega}_{\rm rest}$ to the magnetic field. An obvious one is spin precession. By differentiating $\mathbf{s}$ (19) with respect to proper time, we obtain the precession equation

$$\dot{\mathbf{s}} = \langle \mathbf{\Omega}_R \mathbf{s} \rangle_{\Re} = \boldsymbol{\omega}_R \times \mathbf{s} \tag{20}$$

with $\mathbf{\Omega}_R \equiv 2\dot{R}\bar{R} = -\mathbf{\Omega}_R^\dagger = -i\boldsymbol{\omega}_R$. The spin paravector w in the lab is related to $\mathbf{s}$ by

$$w = \Lambda \mathbf{e}_3 \Lambda^\dagger = B\mathbf{s}B \ ,$$

and the equation of motion for w can be expressed using either the first or second equality:

$$\begin{aligned} \dot{w} = \frac{d}{d\tau}\left(\Lambda \mathbf{e}_3 \Lambda^\dagger\right) &= \langle \mathbf{\Omega} w \rangle_{\Re} = B \langle \mathbf{\Omega}_{\rm rest} \mathbf{s} \rangle_{\Re} B \\ &= \langle \mathbf{\Omega}_B w \rangle_{\Re} + B\dot{\mathbf{s}}B, \end{aligned}$$

where $\mathbf{\Omega}_B \equiv 2\dot{B}\bar{B}$. By now, we have defined a bunch of spacetime rotation rates, but fortunately, they are simply related. Note in particular

$$\begin{aligned} \mathbf{\Omega}_{\rm rest} = \bar{B}\mathbf{\Omega}B = 2\bar{B}\dot{\Lambda}\bar{\Lambda}B &= 2\bar{B}\left(\dot{B}R + B\dot{R}\right)\bar{R} \\ &= 2\left(\bar{B}\dot{B} + \dot{R}\bar{R}\right) = \mathbf{\Omega}_B^\dagger + \mathbf{\Omega}_R \ , \end{aligned} \tag{21}$$

For simplicity, consider first the case that the electric field vanishes in the nonspinning rest frame. According to the Lorentz-force equation,

(17) with (18), in this case, the acceleration $\dot{u}$ vanishes, and consequently so do $\dot{B}$ and $\mathbf{\Omega}_B$. The precession rate is then proportional to $\mathbf{B}_{\rm rest}$, and the g-factor is defined so that it is $-\frac{1}{2}eg\mathbf{B}_{\rm rest}/m$. The full spacetime rotation rate $\mathbf{\Omega}_{\rm rest}$ has only a bivector part

$$\mathbf{\Omega}_{\rm rest} = i\frac{eg}{2m}\mathbf{B}_{\rm rest} - i\omega_0\mathbf{s}, \tag{22}$$

where the first term on the right gives the precession rate of the spin and the second term generates the spin rotation as mentioned above.

More generally, the presence of a rest-frame electric field adds an acceleration $e\mathbf{E}_{\rm rest}/m$ to $\mathbf{\Omega}_{\rm rest}$:

$$\mathbf{\Omega}_{\rm rest} = \frac{e}{m}\left(\mathbf{E}_{\rm rest} + i\frac{g}{2}\mathbf{B}_{\rm rest}\right) - i\omega_0\mathbf{s}\ .$$

Since

$$\mathbf{E}_{\rm rest} = \frac{1}{2}\left(\mathbf{F}_{\rm rest} + \mathbf{F}^\dagger_{\rm rest}\right),\ i\mathbf{B}_{\rm rest} = \frac{1}{2}\left(\mathbf{F}_{\rm rest} - \mathbf{F}^\dagger_{\rm rest}\right)$$

and $\mathbf{F}_{\rm rest} = \bar{B}\mathbf{F}B$, transformation to the lab gives the rotation rate

$$\begin{aligned}\mathbf{\Omega} = B\mathbf{\Omega}_{\rm rest}\bar{B} &= \frac{e}{2m}\left[\mathbf{F} + u\mathbf{F}^\dagger\bar{u} + \frac{g}{2}\left(\mathbf{F} - u\mathbf{F}^\dagger\bar{u}\right)\right] + \omega_0\mathbf{S} \\ &= \frac{e}{4m}\left[(2+g)\,\mathbf{F} + (2-g)\,u\mathbf{F}^\dagger\bar{u}\right] + \omega_0\mathbf{S}\ ,\end{aligned} \tag{23}$$

where from (13) and (19) $\mathbf{S} = -iB\mathbf{s}\bar{B}$. The equation of motion for w is

$$\dot{w} = \langle\mathbf{\Omega}w\rangle_{\Re} = \frac{e}{4m}\left[(2+g)\,\langle\mathbf{F}w\rangle_{\Re} + (2-g)\left\langle u\mathbf{F}^\dagger\bar{u}w\right\rangle_{\Re}\right]\ ,$$

which is exactly the algebraic form of the BMT equation.[6] The term involving $\mathbf{S}$ drops out since $\mathbf{S}w$ is a pure bivector and thus imaginary. If we demand that the eigenspinor evolution (5) is linear in Λ, then $\mathbf{\Omega}$ should not depend on the proper velocity u.[7] Then, from (23), we must have $g = 2$ and the spacetime rotation rate $\mathbf{\Omega}$ is reduces to $e\mathbf{F}/m + \omega_0\mathbf{S}$.

Note that the imaginary part of relation (21) relates the precession rate in the nonspinning rest frame to that in the lab. The difference between the lab and rest-frame precession frequencies is known as the (proper) Thomas precession and is given by

$$\boldsymbol{\omega}_{\rm Th} = i\,\langle\mathbf{\Omega}_R - \mathbf{\Omega}_{\rm rest}\rangle_{\Im} = i\,\langle\mathbf{\Omega}_B\rangle_{\Im} = 2i\left\langle\dot{B}\bar{B}\right\rangle_{\Im}.$$

It is easy to evaluate $\boldsymbol{\omega}_{\rm Th}$ from the expression (14) for B, since the only imaginary contribution $\dot{B}\bar{B}$ arises from the term $\langle\dot{u}\bar{u}\rangle_{\Im}/(2\gamma+2)$:

$$\boldsymbol{\omega}_{\rm Th} = \frac{2i}{2\,(\gamma+1)}\,\langle\dot{u}\bar{u}\rangle_{\Im} = \frac{\dot{\mathbf{u}}\times\mathbf{u}}{\gamma+1}\ . \tag{24}$$

Magnetic Moment

The torque physically responsible for the precession of the spin in a magnetic field arises from the coupling of the magnetic dipole moment of the particle to the field. This coupling can be found independently from the spin rotation rate (22) in the nonspinning rest frame. If we assume a magnetic field $B_{\rm rest} \ll m\omega_0/e$ ($\simeq 4.414 \times 10^9$ tesla when $\omega_0 = 2m/\hbar$), the rotation associated with the intrinsic spin dominates:

$$\begin{aligned} \mathbf{\Omega}_{\rm rest} &= -i\omega_0 \mathbf{s} \left(1 - \frac{e}{m\omega_0} \mathbf{s} \mathbf{B}_{\rm rest} \right) \\ &\simeq -i\omega_0 \mathbf{s}' \left(1 - \frac{e}{m\omega_0} \mathbf{s} \cdot \mathbf{B}_{\rm rest} \right) \end{aligned} \tag{25}$$

with

$$\begin{aligned} \mathbf{s}' = R_1 \mathbf{s} R_1^\dagger &\simeq \mathbf{s} \left(1 - \frac{e}{m\omega_0} \left\langle \mathbf{s} \mathbf{B}_{\rm rest} \right\rangle_V \right) \\ R_1 &= \exp\left(\frac{e}{2m\omega_0} \left\langle \mathbf{s} \mathbf{B}_{\rm rest} \right\rangle_V \right) . \end{aligned}$$

The shift in magnitude of the proper rotation frequency, together with the association $\hbar\omega_0 = 2m$ of that frequency with the mass of the charge, implies a potential energy that shifts the mass. From (25), this potential energy can be expressed

$$-\frac{e\hbar}{2m} \mathbf{s} \cdot \mathbf{B}_{\rm rest} = -\boldsymbol{\mu} \cdot B_{\rm rest} \tag{26}$$

with a magnetic dipole moment that for the electron with $e < 0$ is

$$\boldsymbol{\mu} = \frac{e\hbar}{2m} \mathbf{s} = -g\mu_0 \frac{\mathbf{s}}{2} = -\mu_0 \mathbf{s} \ , \tag{27}$$

where $\hbar\mathbf{s}/2$ is the spin vector, $\mu_0 = |e|\,\hbar/\left(2m\right)$ is the Bohr magneton, and $g = 2$ is the g factor.

5. Dirac Equation

A simple classical equation of motion[7, 8] follows from the Lorentz transformation $p = \Lambda m \Lambda^\dagger$ and the unimodularity of Λ :

$$p\bar{\Lambda}^\dagger = m\Lambda \ . \tag{28}$$

This is the *classical Dirac equation,* which has the form of the quantum Dirac equation in momentum space. It is a real linear equation, in that any real linear combination of solutions Λ is another solution.

Elementary particles are often modeled classically as point charges. An obvious advantage of point charges is that they are simple and structureless. A disadvantage is that their electromagnetic energy is infinite, requires mass renormalization, and leads to preacceleration. An alternative approach is to assume no *a priori* distribution. Instead, a current density $j(x)$ is related by an eigenspinor *field* $\Lambda(x)$ to the reference-frame density ρ_{ref}:

$$j(x) = \Lambda(x)\,\rho_{\text{ref}}(x)\,\Lambda^{\dagger}(x) \ . \tag{29}$$

This form allows the velocity and orientation to be different at different spacetime positions x. The current density (29) can be written in terms of a density-normalized eigenspinor Ψ as

$$j = \Psi\Psi^{\dagger},\ \Psi = \rho_{\text{ref}}^{1/2}\Lambda \ ,$$

where Ψ obeys the same classical Dirac equation as Λ:

$$p\bar{\Psi}^{\dagger} = m\Psi \ . \tag{30}$$

As with the eigenspinor equation of motion (5), this equation (30) is invariant under gauge rotations $\Psi \rightarrow \Psi R$, real linear combinations of solutions are also solutions, and free particle solutions have the form of de Broglie waves

$$\Psi(x) = \Psi(0)\exp\left[-i\mathbf{e}_3\left\langle p\bar{x}\right\rangle_S/\hbar\right] \ . \tag{31}$$

Recall that we used our gauge freedom to choose the spin axis to be $\mathbf{e}_3$. This restricts further gauge rotations to also be about $\mathbf{e}_3$:

$$\Psi \rightarrow \Psi\exp\left(-i\mathbf{e}_3\phi\right), \tag{32}$$

where the scalar parameter ϕ effectively sets the angular position from which rotation angles about $\mathbf{e}_3$ are measured.

If Ψ is assumed to be a real linear superposition of classical de Broglie waves (31), the momentum p can be replaced by a differential operator:

$$p\bar{\Psi}^{\dagger} = i\hbar\partial\bar{\Psi}^{\dagger}\mathbf{e}_3 = m\Psi. \tag{33}$$

The differential form of p has implications for the remaining gauge rotations. If the rotation angle 2ϕ is fixed, the transformation (32) is said to be a *global gauge transformation* and there is no change in the differential form (33) of the Dirac equation. Indeed, we can multiply from the right by any constant element that commutes with $\mathbf{e}_3$. However, if ϕ is a function of position x, then (32) is a *local gauge transformation* and the Dirac equation (33) picks up an additional term:

$$i\hbar\partial\bar{\Psi}^{\dagger}\mathbf{e}_3 + \hbar\left(\partial\phi\right)\bar{\Psi}^{\dagger} = m\Psi.$$

In order to have an equation of motion that is invariant under local gauge transformations, we play the usual game of introducing the paravector gauge potential A as part of the "covariant derivative":

$$p\bar{\Psi}^\dagger = i\hbar\partial\bar{\Psi}^\dagger\mathbf{e}_3 - eA\bar{\Psi}^\dagger = m\Psi, \tag{34}$$

and the gauge transformation (32) is accompanied by a gauge transformation of A :

$$eA \to eA + \hbar\,(\partial\phi)\,.$$

The result (34) is a covariant algebraic form of the usual quantum Dirac equation in APS.

To cast the equation into a form that makes it easier to solve, we split it into two complementary minimal left ideals of APS with the projectors $\mathsf{P}_{\pm 3} \equiv \frac{1}{2}\,(1 \pm \mathbf{e}_3)$:

$$\begin{aligned} p\bar{\Psi}^\dagger\mathsf{P}_{+3} &= (i\hbar\partial - eA)\,\bar{\Psi}^\dagger\mathsf{P}_{+3} = m\Psi\mathsf{P}_{+3} \\ p\bar{\Psi}^\dagger\mathsf{P}_{-3} &= (-i\hbar\partial - eA)\,\bar{\Psi}^\dagger\mathsf{P}_{-3} = m\Psi\mathsf{P}_{-3}\ , \end{aligned}$$

where we noted the "pacwoman" property $\mathbf{e}_3\mathsf{P}_{\pm 3} = \pm\mathsf{P}_{\pm 3}$. We can bar-dagger conjugate the second equation to project it into the same minimal left ideal of the algebra as the first:

$$\bar{p}\Psi\mathsf{P}_{+3} = \left(i\hbar\bar{\partial} - e\bar{A}\right)\Psi\mathsf{P}_{+3} = m\bar{\Psi}^\dagger\mathsf{P}_{+3}\ .$$

Then, defining

$$\psi^{(W)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Psi\mathsf{P}_{+3} \\ \bar{\Psi}^\dagger\mathsf{P}_{+3} \end{pmatrix},$$

we obtain a pair of projected equations in the matrix form

$$\begin{pmatrix} 0 & p \\ \bar{p} & 0 \end{pmatrix}\psi^{(W)} = \begin{pmatrix} 0 & i\hbar\partial - eA \\ i\hbar\bar{\partial} - e\bar{A} & 0 \end{pmatrix}\psi^{(W)} = m\psi^{(W)}. \tag{35}$$

which is fully equivalent to our single algebraic equation (34). The projector P_3 gobbles $\mathbf{e}_3$ and changes the spin rotor $\exp\left(-i\omega_0\tau\mathbf{e}_3/2\right)$ into a simple phase factor. If the standard matrix representation $\mathbf{e}_\mu \to \underline{\sigma}_\mu$ is used for APS in terms of the Pauli spin matrices $\underline{\sigma}_\mu$, Eq. (35) is the traditional form[9] of the Dirac equation $p^\mu\gamma_\mu\psi^{(W)} = m\psi^{(W)}$ with gamma matrices in the Weyl (or spinor) representation. The projection of the algebraic Ψ by $\mathsf{P}_{\pm 3}$ picks out the upper and lower components of $\psi^{(W)}$ and is seen to be equivalent to multiplication of $\psi^{(W)}$ by the traditional chirality projectors $\frac{1}{2}\,(1 \pm \gamma_5)$ with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$:

$$\frac{1}{2}\,(1 \pm \gamma_5)\,\psi^{(W)} \Leftrightarrow \Psi\mathsf{P}_{\pm 3}\ .$$

The Dirac bispinor in the Dirac-Pauli (or standard) representation is related by

$$\psi^{(DP)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi^{(W)} = \begin{pmatrix} \langle \Psi \rangle_+ \mathsf{P}_{+3} \\ \langle \Psi \rangle_- \mathsf{P}_{+3} \end{pmatrix}, \tag{36}$$

where $\langle \Psi \rangle_\pm = \frac{1}{2} \left(\Psi \pm \bar{\Psi}^\dagger \right)$ are the even and odd parts of Ψ and correspond to the *large* and *small* components at low velocities:

$$\begin{aligned} \langle \Psi \rangle_+ &= \rho_{\text{ref}}^{1/2} \langle B \rangle_+ R = \rho_{\text{ref}}^{1/2} \sqrt{\frac{m+E}{2m}} R \simeq \rho_{\text{ref}}^{1/2} R \\ \langle \Psi \rangle_- &= \rho_{\text{ref}}^{1/2} \langle B \rangle_- R = \frac{\mathbf{p}}{m+E} \langle \Psi \rangle_+ \ . \end{aligned}$$

The last expression for $\langle \Psi \rangle_+$ on the RHS is the low-velocity approximation. In the rest frame, the small component disappears and the eigenfunction is even. We say the particle has even *intrinsic parity.* More generally, the solutions $\psi^{(W)}$ (35) and $\psi^{(DP)}$ (36) are 4×2 matrices whose second columns are zero and whose first columns give the usual Dirac bispinors of quantum theory. If R is replaced by the de Broglie spin rotor $R = \exp\left[-i\mathbf{e}_3 \langle p\bar{x} \rangle_S / \hbar\right]$, the solutions in the nonvanishing columns of $\psi^{(W)}$ and $\psi^{(DP)}$ are the usual momentum eigenstates of the Dirac equation.

Spin Distributions

To study spin distributions, the low-velocity limit

$$\Psi \simeq \langle \Psi \rangle_+ \simeq \rho_{\text{ref}}^{1/2} R \tag{37}$$

is sufficient. In terms of Euler angles ϕ, θ, χ, about space-fixed axes, the rotor R can be expressed by

$$R = \exp\left(-i\mathbf{e}_3 \phi/2\right) \exp\left(-i\mathbf{e}_2 \theta/2\right) \exp\left(-i\mathbf{e}_3 \chi/2\right). \tag{38}$$

The classical spin direction in a static system is $\mathbf{s} = R\mathbf{e}_3 R^\dagger$ [see (19) above], where R may be a function of position $\mathbf{r}$. A distribution of such spin directions is thus $\rho_{\text{ref}} R\mathbf{e}_3 R^\dagger$, where the positive scalar $\rho_{\text{ref}} = \rho_{\text{ref}}(\mathbf{r})$ is the density of spins in the reference frame. As seen below, simple measurements of the spin direction give only one component at a time. The distribution of the component of the spin in the direction of an arbitrary unit vector $\mathbf{m}$ is

$$\rho_{\text{ref}}\ \mathbf{s} \cdot \mathbf{m} = \left\langle \rho_{\text{ref}} R\mathbf{e}_3 R^\dagger \mathbf{m} \right\rangle_S \ .$$

In terms of the projector P_{+3}, since $\mathbf{e}_3 = \mathsf{P}_{+3} - \mathsf{P}_{-3}$ and for any elements p, q, $\langle pq \rangle_S = \langle qp \rangle_S = \langle \overline{pq} \rangle_S$, the distribution is

$$\begin{aligned} 2 \left\langle \rho_{\text{ref}} R \mathsf{P}_{+3} R^\dagger \mathbf{m} \right\rangle_S &= 2 \left\langle \mathsf{P}_{+3} R^\dagger \rho_{\text{ref}}^{1/2} \mathbf{m} \rho_{\text{ref}}^{1/2} R \mathsf{P}_{+3} \right\rangle_S \\ &= \text{tr} \left\{ \psi^\dagger \mathbf{m} \psi \right\} , \end{aligned} \tag{39}$$

where by ψ we mean the ideal spinor, whose standard matrix representation is

$$\psi \equiv \rho_{\text{ref}}^{1/2} R \mathsf{P}_{+3} = e^{-i\chi/2} \rho_{\text{ref}}^{1/2} \begin{pmatrix} e^{-i\phi/2} \cos\theta/2 & 0 \\ e^{i\phi/2} \sin\theta/2 & 0 \end{pmatrix} .$$

If the column of zeros is dropped, the standard matrix representation of ψ is a two-component spinor, as familiar from the usual nonrelativistic Pauli theory. Such spinors carry an irreducible representation of the rotation group $SU(2)$.[2] It may be noted that the full rotor R can always be recovered from the ideal projection $R\mathrm{P}_3$ as twice its even part:

$$R = 2 \langle R \mathrm{P}_3 \rangle_+ \ .$$

The term $\psi^\dagger \mathbf{m} \psi$ is a scalar and tr can be omitted from (39). Although we derived the spin distribution as a classical expression, it has *precisely the quantum form* if we recognize that the matrix representation of the unit vector $\mathbf{m}$, namely $\mathbf{m} = m^j \mathbf{e}_j \rightarrow m^1 \sigma_x + m^2 \sigma_y + m^3 \sigma_z$, is traditionally (but misleadingly, since it represents a vector, not a scalar) written $\mathbf{m} \cdot \boldsymbol{\sigma}$. From the definition of ρ_{ref} the spinor ψ satisfies the usual normalization condition,

$$2 \int d^3x \ \left\langle \psi^\dagger \psi \right\rangle_S \equiv \langle \psi | \psi \rangle = 1$$

and the average component of the spin in the direction $\mathbf{m}$ is

$$2 \int d^3x \ \left\langle \psi^\dagger \mathbf{m} \psi \right\rangle_S \equiv \langle \psi \, | \mathbf{m} | \, \psi \rangle .$$

From expression (39) we see that the real paravector $\mathsf{P}_\mathbf{s} = R \mathsf{P}_{+3} R^\dagger = \frac{1}{2}(1 + \mathbf{s})$ embodies information about the classical spin state at a given point in space or in a homogeneous ensemble. It is equivalent to the quantum spin density operator $\varrho \sim \psi \psi^\dagger$ for the pure state of spin $\mathbf{s}$,

[2] More generally, once projected onto a minimal left ideal, any spinor $\Psi = \rho_{\text{ref}}^{1/2} B R$ is equivalent to a dilated spatial rotation, thereby reducing this representation of the noncompact Lorentz group to a scaling factor times the compact group $SU(2)$.

and it is also a projector that acts as a state filter. The component of the spin in the $\mathbf{m}$ direction is $2\langle \mathsf{P}_{\mathbf{s}}\,\mathbf{m}\rangle_S$, whose matrix representation is identical to the usual quantum expression, traditionally written $\frac{1}{2}\mathrm{tr}\left\{\varrho\,\mathbf{m}\cdot\sigma\right\}$. Note that the part of a rotation R around the spin axis becomes a phase factor of $\psi = \rho_{\mathrm{ref}}^{1/2} R\mathsf{P}_{+3}$, since the pacwoman property gives $e^{-i\chi\mathbf{e}_3/2}\mathsf{P}_{+3} = e^{-i\chi/2}\mathsf{P}_{+3}$. Thus, when phase factors of ψ are ignored, information about the axial rotation of R is lost. Good quantum calculations keep track of *relative* phase, and this appears to be the only aspect of the phase (rotation about $\mathbf{e}_3$) that can be determined experimentally.

One way of seeing whether the system is in a given state of spin $\mathbf{n}$ is to apply the state filter to the spin density operator ϱ and see what remains:

$$\mathsf{P}_{\mathbf{n}}\varrho\mathsf{P}_{\mathbf{n}} = \left(\mathsf{P}_{\mathbf{n}}\varrho + \bar{\varrho}\bar{\mathsf{P}}_{\mathbf{n}}\right)\mathsf{P}_{\mathbf{n}} = 2\left\langle \mathsf{P}_{\mathbf{n}}\varrho\right\rangle_S \mathsf{P}_{\mathbf{n}}\,. \tag{40}$$

The scalar coefficient $2\left\langle \mathsf{P}_{\mathbf{n}}\varrho\right\rangle_S = \left\langle (1+\mathbf{n})\,\varrho\right\rangle_S$ is the probability of finding the system described by ϱ in the state $\mathbf{n}$. For a system in the pure state $\varrho = \mathsf{P}_{\mathbf{s}} = \frac{1}{2}(1+\mathbf{s})$, the probability is

$$2\left\langle \mathsf{P}_{\mathbf{n}}\mathsf{P}_{\mathbf{s}}\right\rangle_S = \frac{1}{2}\left\langle (1+\mathbf{n})\,(1+\mathbf{s})\right\rangle_S = \frac{1}{2}\,(1+\mathbf{n}\cdot\mathbf{s})\,. \tag{41}$$

This is unity if the system is definitely in the state $\mathbf{n}$, whereas it vanishes if the system is in a state *orthogonal* to $\mathbf{n}$. Thus, $\mathbf{s}=\mathbf{n}$ is required for the states to be the same and $\mathbf{s}=-\mathbf{n}$ for the states to be orthogonal. Note that the mathematics is the same as used in my first lecture to describe light polarization.[1]

Spin $\frac{1}{2}$ and State Expansions

The value of $\frac{1}{2}$ for the spin of elementary spinors considered here arises in several ways. It is the group-theoretical label for the irreducible spinor representation of the rotation group $SU(2)$ carried by ideal spinors. It is also required by the fact that any rotation can be expressed as a linear superposition of two independent orthogonal rotations defined for any direction in space. The Euler-angle form (38) of any rotor R can be rewritten

$$\begin{aligned} R &= \exp\left(-i\mathbf{n}\theta/2\right)\exp\left[-i\mathbf{e}_3\left(\phi+\chi\right)/2\right] \\ &= \left(\cos\frac{\theta}{2} - i\mathbf{n}\sin\frac{\theta}{2}\right)\exp\left[-i\mathbf{e}_3\left(\phi+\chi\right)/2\right], \end{aligned} \tag{42}$$

where $\mathbf{n} = \exp\left(-i\mathbf{e}_3\phi/2\right)\mathbf{e}_2\exp\left(i\mathbf{e}_3\phi/2\right)$ is a unit vector in the $\mathbf{e}_1\mathbf{e}_2$ plane. Therefore, any rotor R is a real linear combination $\cos\frac{\theta}{2}R_{\uparrow} +$

$\sin\frac{\theta}{2}R_\downarrow$ of rotors $R_\uparrow = \exp\left[-i\mathbf{e}_3\left(\phi+\chi\right)/2\right]$ and $R_\downarrow = -i\mathbf{n}R_\uparrow$ that are mutually orthogonal: $\left\langle R_\uparrow R_\downarrow^\dagger \right\rangle_S = \langle -i\mathbf{n}\rangle_S = 0$.

By projecting the rotors with P_{+3} we obtain the equivalent relation of ideal spinors:

$$\begin{aligned} R\mathrm{P}_{+3} &= \cos\frac{\theta}{2}R_\uparrow\mathrm{P}_{+3} + \sin\frac{\theta}{2}R_\downarrow\mathrm{P}_{+3} \\ \psi \equiv \rho_{\mathrm{ref}}^{1/2}R\mathsf{P}_{+3} &= \cos\frac{\theta}{2}\psi_\uparrow + \sin\frac{\theta}{2}\psi_\downarrow \\ \psi_\uparrow = \rho_{\mathrm{ref}}^{1/2}e^{-i(\phi+\chi)/2}\mathrm{P}_{+3},\ \psi_\downarrow &= -i\mathbf{n}\psi_\uparrow\ . \end{aligned} \tag{43a}$$

Traditional orthonormality conditions hold:

$$\begin{aligned} 2\left\langle \psi_\uparrow\psi_\downarrow^\dagger \right\rangle_S &= 2\left\langle \psi_\uparrow\psi_\uparrow^\dagger i\mathbf{n} \right\rangle_S = 2\rho_{\mathrm{ref}}\left\langle \mathrm{P}_{+3}i\mathbf{n}\right\rangle_S = 0 \\ 2\left\langle \psi_\downarrow\psi_\downarrow^\dagger \right\rangle_S &= 2\left\langle \psi_\uparrow\psi_\uparrow^\dagger \right\rangle_S = 2\rho_{\mathrm{ref}}\left\langle \mathrm{P}_{+3}\right\rangle_S = \rho_{\mathrm{ref}}\ . \end{aligned}$$

It follows that the amplitudes are

$$\begin{aligned} \langle\psi_\uparrow|\psi\rangle &= 2\int d^3x\ \left\langle \psi\psi_\uparrow^\dagger \right\rangle_S = \cos\frac{\theta}{2} \\ \langle\psi_\downarrow|\psi\rangle &= 2\int d^3x\ \left\langle \psi\psi_\downarrow^\dagger \right\rangle_S = \sin\frac{\theta}{2} \end{aligned}$$

giving probabilities as found above in Eq. (41).

$$\left|\langle\psi_\uparrow|\psi\rangle\right|^2 = \cos^2\frac{\theta}{2} = \frac{1}{2}\left(1+\mathbf{s}\cdot\mathbf{e}_3\right) \tag{44}$$

$$\left|\langle\psi_\downarrow|\psi\rangle\right|^2 = \sin^2\frac{\theta}{2} = \frac{1}{2}\left(1-\mathbf{s}\cdot\mathbf{e}_3\right). \tag{45}$$

By tacking on an additional fixed rotation, the treatment can be extended to measurements along an arbitrary axis.

Stern-Gerlach Experiment

The basic measurement of spin is that of the Stern-Gerlach experiment [10], in which a beam of ground-state silver atoms is split by a magnetic-field gradient into distinct beams of opposite spin polarization. It is a building block of real and thought experiments in quantum measurement [11].

Consider a nonrelativistic beam of ground-state atoms that travels with velocity $\mathbf{v} = v_x\mathbf{e}_1$ through a static magnetic field $\mathbf{B}$ that vanishes everywhere except in the vicinity of the Stern-Gerlach magnet. The net

effect of the magnet on the beam is a vertical force proportional to the z component μ_z of the magnetic dipole moment, which we take to be the magnetic dipole moment of an electron (27), corresponding to an atomic ground state with an unpaired electron in an S state:

$$\mu_z \frac{\partial B_z}{\partial z} = -\mu_0 \mathbf{s} \cdot \mathbf{e}_3 \frac{\partial B_z}{\partial z} \, .$$

We assume the beam is uniform along its length with $\rho_{\rm ref} = 1$ and that $\partial B_z/\partial z < 0$. The eigenspinor for the initial beam in the nonrelativistic limit is then

$$\Psi = BR \simeq \left(1 + \frac{1}{2}\mathbf{v}\right) R \, .$$

The key point is that, as seen above (42), every rotor is a linear combination of "up" and "down" rotors with opposite spin directions. For a state in which $\mathbf{s}$ makes an angle θ with respect to $\mathbf{e}_3$, $R = \cos\frac{\theta}{2} R_\uparrow + \sin\frac{\theta}{2} R_\downarrow$ with

$$R_\uparrow \mathbf{e}_3 R_\uparrow^\dagger = \mathbf{e}_3, \; R_\downarrow \mathbf{e}_3 R_\downarrow^\dagger = -\mathbf{e}_3 \, . \tag{46}$$

The components of the spin unit vector

$$\mathbf{s} \cdot \mathbf{e}_3 = \left\langle R \mathbf{e}_3 R^\dagger \mathbf{e}_3 \right\rangle_S = \begin{cases} +1, & R = R_\uparrow \\ -1, & R = R_\downarrow \end{cases}$$

are also opposite for the rotors $R_\uparrow$ and $R_\downarrow$ so that the Stern-Gerlach magnet forces the eigenspinor components $R_\uparrow$ and $R_\downarrow$ of the beam eigenspinor upwards and downwards, respectively. This splits the incident beam into two isolated branches, analogous to the way a birefringent crystal splits a beam of light into two branches of orthogonal polarization. The fraction of the initial beam in the upper branch is $\cos^2\theta/2 = \frac{1}{2}(1 + \cos\theta)$ whereas in the lower branch the fraction is $\sin^2\theta/2 = \frac{1}{2}(1 - \cos\theta)$, just as found from the square amplitudes (44) and (45) above. The two-valued property of the measurement is a direct result of the decomposition of any rotor R into rotors for rotations about any two opposite directions, rotations that correspond to "spin-up" and "spin-down" components.

These relations can be re-expressed in the form of eigenspinor ideals, which is closer to the quantum formulation. Since the rotors $R_\uparrow, R_\downarrow$ are unitary, relation (46) can be cast in a form

$$\mathbf{e}_3 R_\uparrow = R_\uparrow \mathbf{e}_3, \; \mathbf{e}_3 R_\downarrow = -R_\downarrow \mathbf{e}_3$$

that becomes an eigenfunction equation when projected onto the P_{+3} ideals $\psi_{\uparrow\downarrow} = R_{\uparrow\downarrow}\mathrm{P}_3$:

$$\mathbf{e}_3 \psi_\uparrow = \psi_\uparrow, \; \mathbf{e}_3 \psi_\downarrow = -\psi_\downarrow \, .$$

Evidently $\mathbf{e}_3 = \mathrm{P}_3 - \bar{\mathrm{P}}_3$ is the spin operator corresponding to the z component $\mathbf{s}\cdot\mathbf{e}_3$ of the spin in units of $\hbar/2$ for the P_3 ideal representation of ψ. The projected rotor is split into $R\mathrm{P}_3 = \psi = \cos\frac{\theta}{2}\psi_\uparrow + \sin\frac{\theta}{2}\psi_\downarrow$, the two parts of which experience opposite forces from the Stern-Gerlach magnet.

6. Bell's Theorem

Quantum systems are often tested against Bell's theorem. Violation of Bell's inequalities is generally held to be evidence of nonlocal quantum weirdness. A simple proof of the inequalities can be derived from polarization splitters acting on a single beam. The nice thing about this derivation is that the question of locality vs. nonlocality does not arise. It shows that Bell's inequality concerns directly the difference between classical probabilities for particles and quantum probability amplitudes in waves. Nonlocality arises only in EPR-type applications with entangled states, but the inequality itself is violated locally for a single beam of waves.

Consider a beam of silver atoms passing through a sequence of Stern-Gerlach magnets aligned to split the beam into spins polarized in opposite directions specified by unit vectors $\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{c}$. If one of the polarized beams leaving a given Stern-Gerlach magnet is blocked, that magnet serves as a polarization filter. Let $f(\mathbf{c},\mathbf{b},\mathbf{a})$ be the fraction of atoms passing through sequential polarization filters of types $\mathbf{a},\mathbf{b},\mathbf{c}$. Then, with "classical" atoms (the same math works for polarized photons), the fraction that pass through filters $\mathbf{a}$ and then $\mathbf{c}$ is just the sum of those that pass through $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{a},-\mathbf{b},\mathbf{c}$:

$$f(\mathbf{c},\mathbf{a}) = f(\mathbf{c},\mathbf{b},\mathbf{a}) + f(\mathbf{c},-\mathbf{b},\mathbf{a})\,.$$

Similarly,

$$f(-\mathbf{c},\mathbf{b}) = f(-\mathbf{c},\mathbf{b},\mathbf{a}) + f(-\mathbf{c},\mathbf{b},-\mathbf{a})$$
$$f(\mathbf{c},\mathbf{b},\mathbf{a}) + f(-\mathbf{c},\mathbf{b},\mathbf{a}) = f(\mathbf{b},\mathbf{a})$$

Adding these three relations we obtain

$$\begin{aligned} f(\mathbf{c},\mathbf{a}) + f(-\mathbf{c},\mathbf{b}) &= f(\mathbf{b},\mathbf{a}) + f(\mathbf{c},-\mathbf{b},\mathbf{a}) + f(-\mathbf{c},\mathbf{b},-\mathbf{a}) \\ &\geq f(\mathbf{b},\mathbf{a}) \end{aligned} \tag{47}$$

which is a form of Bell's inequality. We found above (41) that $f(\mathbf{c},\mathbf{a}) = \frac{1}{2}(1+\mathbf{c}\cdot\mathbf{a})$ so that Bell's inequality (47) can be written

$$1 + \mathbf{c}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{b} \geq \mathbf{b}\cdot\mathbf{a}\,.$$

If we let $\mathbf{b}$ bisect $\mathbf{a}$ and $\mathbf{c}$, so that we can put $\mathbf{c}\cdot\mathbf{a} = \cos 2\theta$ and $\mathbf{c}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \cos\theta$, we obtain

$$1 + \cos 2\theta \geq 2\cos\theta.$$

However, this inequality is broken for $\theta = \pi/4$, for example. The error is in working with fractions or probabilities instead of amplitudes. Using our filter method (40), we find indeed that

$$\begin{aligned} f(\mathbf{c},\mathbf{a}) &= 2\langle \mathrm{P}_{\mathbf{c}}\mathrm{P}_{\mathbf{a}}\mathrm{P}_{\mathbf{c}}\rangle_S = \frac{1}{2}(1+\mathbf{a}\cdot\mathbf{c}) \\ &= 2\left\langle \mathrm{P}_{\mathbf{c}}\left(\mathrm{P}_{\mathbf{b}}+\bar{\mathrm{P}}_{\mathbf{b}}\right)\mathrm{P}_{\mathbf{a}}\left(\mathrm{P}_{\mathbf{b}}+\bar{\mathrm{P}}_{\mathbf{b}}\right)\mathrm{P}_{\mathbf{c}}\right\rangle_S \ , \end{aligned}$$

but this is not the same as

$$\begin{aligned} f(\mathbf{c},\mathbf{b},\mathbf{a}) + f(\mathbf{c},-\mathbf{b},\mathbf{a}) &= 2\langle \mathrm{P}_{\mathbf{c}}\mathrm{P}_{\mathbf{b}}\mathrm{P}_{\mathbf{a}}\mathrm{P}_{\mathbf{b}}\mathrm{P}_{\mathbf{c}}\rangle_S + 2\left\langle \mathrm{P}_{\mathbf{c}}\bar{\mathrm{P}}_{\mathbf{b}}\mathrm{P}_{\mathbf{a}}\bar{\mathrm{P}}_{\mathbf{b}}\mathrm{P}_{\mathbf{c}}\right\rangle_S \\ &= \langle \mathrm{P}_{\mathbf{c}}\mathrm{P}_{\mathbf{b}}\mathrm{P}_{\mathbf{c}}\rangle_S (1+\mathbf{b}\cdot\mathbf{a}) + \left\langle \mathrm{P}_{\mathbf{c}}\bar{\mathrm{P}}_{\mathbf{b}}\mathrm{P}_{\mathbf{c}}\right\rangle_S (1-\mathbf{b}\cdot\mathbf{a}) \\ &= \frac{1}{2}(1+\mathbf{b}\cdot\mathbf{c}\ \mathbf{b}\cdot\mathbf{a}) \end{aligned}$$

unless $\mathbf{b} = \mathbf{a}$ or $\mathbf{b} = \mathbf{c}$.

7. Qubits and Entanglement

The basis of quantum computation[12] lies in the replacement of the classical bit, with its discrete binary values $(1, 0)$ [for (on, off) or (true, false)], by a two-level quantum system, called a quantum bit or *qubit*, that can generally exist in a superposition of its two states. The unitary evolution of a coupled system of qubits in an arbitrary superposition of states simultaneously computes the evolution of all binary sequences in the superposition, and this massive quantum parallelism offers huge advantages for certain problems, such as factoring large numbers and some types of large searches.

The simplest example of a two-level quantum system is the spin-1/2 system considered above. The same formalism can be applied to qubits. Here, we mainly formulate the algebra in the minimum left ideal $C\ell_3\mathrm{P}_3$ of APS and tensor products thereof, since these lie closest to the quantum formalism. We first consider operations on single spin systems in the language of qubits and quantum computing, and next present some extensions to systems of two or more qubits.

Qubit Operators

There is arbitrariness in the phase as well as in factors of $\mathbf{e}_3$ in $\psi_\uparrow$ and $\psi_\downarrow$. Let us therefore define a *standard ideal spinor basis,* which we

denote in the ket notation

$$\begin{aligned}|\uparrow\rangle &= \mathrm{P}_3 \\ |\downarrow\rangle &= \mathbf{e}_1 |\uparrow\rangle = \mathbf{e}_1\mathrm{P}_3 |\uparrow\rangle = \mathbf{e}_1\mathbf{e}_3\mathrm{P}_3 |\uparrow\rangle = -i\mathbf{e}_2\mathrm{P}_3 |\uparrow\rangle \ .\end{aligned}$$

The *unipotent* ($\mathbf{e}_1^2 = 1$) element $\mathbf{e}_1$interchanges these: $|\downarrow\rangle = \mathbf{e}_1 |\uparrow\rangle$ and $|\uparrow\rangle = \mathbf{e}_1 |\downarrow\rangle$. We may think of the "up" state $|\uparrow\rangle$ as representing the bit value 1, and the "down" state $|\downarrow\rangle$ as the bit value 0. Evidently, $\mathbf{e}_1$ acts as the NOT operator. It is also a *reflection operator*, since for any vector $\mathbf{v}$

$$\mathbf{v} \to \mathbf{e}_1\mathbf{v}\mathbf{e}_1 = \mathbf{v}^{\parallel} - \mathbf{v}^{\perp} = 2\mathbf{v}\cdot\mathbf{e}_1\ \mathbf{e}_1 - \mathbf{v}$$

keeps the component $\mathbf{v}^{\parallel}$ of $\mathbf{v}$ along $\mathbf{e}_1$ constant and reverses components perpendicular to $\mathbf{e}_1$. The dual to $\mathbf{e}_1$ is $-i\mathbf{e}_1 = \exp(-i\pi\mathbf{e}_1/2)$, a rotor for the rotation of π about $\mathbf{e}_1$. If it is applied twice, the result is a 2π rotation which changes the sign of any spinor for a spin-1/2 system: $(-i\mathbf{e}_1)^2 = -1$. Note that any unit vector in the $\mathbf{e}_{12}$ plane can serve as the NOT operator. The resultant states differ only in their relative phase.

A state in which the spin is aligned with $\mathbf{e}_1$ is an eigenstate of the NOT operator. Such a state is obtained by applying the rotor $\exp(-i\pi\mathbf{e}_2/4)$ to P_3, which rotates the spin from $\mathbf{e}_3$ by $\pi/2$ in the $\mathbf{e}_3\mathbf{e}_1$ plane. It can be expressed in several ways:

$$e^{-i\pi\mathbf{e}_2/4}\mathrm{P}_3 = \left(\frac{1 - i\mathbf{e}_2}{\sqrt{2}}\right)\mathrm{P}_3 = \left(\frac{1 + \mathbf{e}_1}{\sqrt{2}}\right)\mathrm{P}_3 = \left(\frac{\mathbf{e}_3 + \mathbf{e}_1}{\sqrt{2}}\right)\mathrm{P}_3 \ .$$

In the fourth form, we see the eigenstate of the NOT operator is also obtained by a reflection of the up basis state in the unit vector $\hat{\mathbf{h}} = (\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{2} = e^{i\pi\mathbf{e}_2/4}\mathbf{e}_1 = e^{-i\pi\mathbf{e}_2/4}\mathbf{e}_3$. This reflection is called the *Hadamard transformation*.

Since any state can be expressed as a linear combination of the basis states P_3 and $\mathbf{e}_1\mathrm{P}_3$, we can find the effect of an operation on any state if we know its effect of the basis states. In the Hadamard transformation,

$$\begin{aligned}|\uparrow\rangle \to \hat{\mathbf{h}}\,|\uparrow\rangle &= \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \\ |\downarrow\rangle \to \hat{\mathbf{h}}\,|\downarrow\rangle &= \hat{\mathbf{h}}\mathbf{e}_1\,|\uparrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} \ .\end{aligned}$$

Exponential Forms

Rotations can be written in exponential form with a scalar parameter proportional to the angle of rotation. They are thus connected to

unity (scalar parameter = 0) and can easily be interpolated. Reflections, on the other hand, are usually viewed as discrete transformations, performed entirely or not at all. However, unit vectors can also be extended to exponential form. Such exponential forms are useful for working out physical interactions in the Hamiltonian that induce the reflections through the time evolution operator.

Consider first the dual $-i\mathbf{n}$ of the real unit vector $\mathbf{n}$. It is a rotor for a rotation by angle π about the $\mathbf{n}$ axis and can be viewed as the continuous transformation $\exp(-i\mathbf{n}\alpha/2)$ when $\alpha = \pi$:

$$-i\mathbf{n} = \exp(-i\mathbf{n}\alpha/2)\,,\ \alpha = \pi\ .$$

Thus the unit vector $\mathbf{n}$ can be expressed

$$\begin{aligned}\mathbf{n} &= i\exp(-i\mathbf{n}\pi/2) = \exp[i(1-\mathbf{n})\pi/2] \\ &= \exp\left[i\bar{\mathrm{P}}_{\mathbf{n}}\pi\right] = \exp\left[i\bar{\mathrm{P}}_{\mathbf{n}}\alpha\right]\,,\ \alpha = \pi\ .\end{aligned}$$

In other words, it is part of a continuous transformation generated by the projector $\bar{\mathrm{P}}_{\mathbf{n}}$. At intermediate angles α we note that

$$e^{i\alpha\bar{\mathrm{P}}} = e^{i\alpha\bar{\mathrm{P}}}\left(\mathrm{P}+\bar{\mathrm{P}}\right) = \mathrm{P} + e^{i\alpha}\bar{\mathrm{P}}\ .$$

Raising and Lowering Operators

We saw above that in the ideal-spinor representation, the unit vector $\mathbf{n}$ represents the spin component in the direction $\mathbf{n}$ in units of $\hbar/2$. One may verify that the commutation relations of unit vectors in space are just what one needs for the spin components, and of course that is why the Pauli spin matrices were introduced to represent electron spin. The operator for the squared spin is thus

$$S^2 = \left(\frac{\hbar}{2}\right)^2\left(\mathbf{e}_1^2+\mathbf{e}_2^2+\mathbf{e}_3^2\right) = \frac{3}{4}\hbar^2.$$

The *raising* operator s_+ is defined by $s_+\left|\downarrow\right\rangle = \left|\uparrow\right\rangle$, $s_+\left|\uparrow\right\rangle = 0$. Any non-null element that annihilates $\mathrm{P}_3 = \left|\uparrow\right\rangle$ must have the form $x\bar{\mathrm{P}}_3$, and

$$x\bar{P}_3\left|\downarrow\right\rangle = x\bar{P}_3\mathbf{e}_1P_3 = x\mathbf{e}_1\left|\uparrow\right\rangle = \left|\uparrow\right\rangle$$

is satisfied by $x = \mathbf{e}_1$. Thus, we can put

$$s_+ = \mathbf{e}_1\bar{\mathrm{P}}_3 = \frac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2) = \mathrm{P}_3\mathbf{e}_1\ . \tag{48}$$

Similarly, the lowering operator is

$$s_- = \mathbf{e}_1\mathrm{P}_3 = s_+^\dagger. \tag{49}$$

The operators $s_\pm$ are examples of nilpotent operators: $s_\pm^2 = 0$. Their products are

$$s_+ s_- = \mathrm{P}_3, \;\; s_- s_+ = \bar{\mathrm{P}}_3 \;,$$

which gives $s_+ s_- + s_- s_+ = 1$ and $s_+ s_- - s_- s_+ = \mathbf{e}_3$.

Two-Qubit Systems

We represent multi-qubit systems very generally as tensor products of single qubit systems, that is, in tensor products of ideals of APS: $\mathcal{C}\ell_3\mathrm{P}_3 \otimes \mathcal{C}\ell_3\mathrm{P}_3$. This representation is sufficiently general to handle distinguishable quantum systems. If the systems are identical, additional symmetries apply. Tensor (or "Kronecker") products obey the fundamental relation

$$(a \otimes b)\,(c \otimes d) = ac \otimes bd$$

Label two qubits A and B. With two qubits or spin-1/2 systems, the projector is the tensor (or "Kronecker") product $\mathrm{P}_3 \otimes \mathrm{P}_3$ and the general pure state can be written

$$\psi_{AB} = \Psi_{AB}\,(\mathrm{P}_3 \otimes \mathrm{P}_3)\,.$$

If Ψ_{AB} is a *product state,* it can be expressed as a single product term $\Psi_{AB} = \Psi_A \otimes \Psi_B$ so that

$$\psi_{AB} = \Psi_{AB}\,(\mathrm{P}_3 \otimes \mathrm{P}_3) = (\Psi_A \otimes \Psi_B)\,(\mathrm{P}_3 \otimes \mathrm{P}_3) = \Psi_A \mathrm{P}_3 \otimes \Psi_B \mathrm{P}_3$$

and the density operator is also a single product

$$\begin{aligned}
\varrho_{AB} = \psi_{AB}\psi_{AB}^\dagger &= (\Psi_A \otimes \Psi_B)\,(\mathrm{P}_3 \otimes \mathrm{P}_3)\,(\Psi_A \otimes \Psi_B)^\dagger \\
&= (\Psi_A \otimes \Psi_B)\,(\mathrm{P}_3 \otimes \mathrm{P}_3)\left(\Psi_A^\dagger \otimes \Psi_B^\dagger\right) \\
&= \left(\Psi_A \mathrm{P}_3 \Psi_A^\dagger\right) \otimes \left(\Psi_B \mathrm{P}_3 \Psi_B^\dagger\right) \\
&= \varrho_A \otimes \varrho_B \;.
\end{aligned}$$

Such systems, whose density operators can be expressed as a single tensor product of subsystem parts, are *decomposable.* In a decomposable system, measurements in one subsystem are independent of those in another. There is no entanglement and indeed no correlation between the subsystems. A system whose density operator is a mixture of decomposable density operators

$$\varrho_{AB} = \sum_j w_j \varrho_A^j \otimes \varrho_B^j$$

is called *disentangled* or *separable.* An entangled state is not separable.

A *reduced density operator* for one subsystem of a composite system is found by summing over all the states of the other subsystems; this is equivalent to tracing over the other subsystems. We can write

$$\varrho_{A\bar{B}} = \mathrm{tr}_B \left\{ \varrho_{AB} \right\}$$

for the reduced density matrix of subsystem A after summing over all states of subsystem B. In the case of a product state $\varrho_{AB} = \varrho_A \otimes \varrho_B$,

$$\varrho_{A\bar{B}} = \mathrm{tr}_B \left\{ \varrho_A \otimes \varrho_B \right\} = \varrho_A \otimes \mathrm{tr}_B \left\{ \varrho_B \right\} = \varrho_A \ .$$

It is often stated that measurements in one subsystem *affect* the outcome of those in another, but such language is misleading and based on classical probability interpretations. It suggests a cause-and-effect relation that propagates faster than the speed of light, and such propagation is not physically possible by relativistic causality. In an entangled system, if measurements on subsystem A are only recorded when a given measurement result is obtained on subsystem B, then the results on A will depend on the measurement outcome selected on B. However, if results are recorded for A regardless of the measured value for B, then the A results cannot depend on the actual measurement performed on B. Mathematically, the density operator for A in that case is the subtrace of ϱ over B, and that subtrace is independent of basis and therefore independent of which property eigenstates are used.

Arbitrary states, including entangled (and hence inseparable) ones, can be expressed as linear combinations of product states:

$$\Psi_{AB} \left(\mathrm{P}_3 \otimes \mathrm{P}_3 \right) = \sum_{j,k} c_{jk} \left[\Psi_A \left(\mathbf{\Omega}_j \right) \otimes \Psi_B \left(\mathbf{\Omega}_k \right) \right] \left(\mathrm{P}_3 \otimes \mathrm{P}_3 \right)$$

The corresponding density operator is

$$\varrho_{AB} = \Psi_{AB} \left(\mathrm{P}_3 \otimes \mathrm{P}_3 \right) \Psi_{AB}^{\dagger} \ .$$

The average value of spin A in the direction $\hat{\mathbf{a}}$ and, simultaneously, spin B in the direction $\hat{\mathbf{b}}$ is $2^2 \left\langle \varrho_{AB} \hat{\mathbf{a}} \otimes \hat{\mathbf{b}} \right\rangle_{S \oplus S}$ where the subscript $S \oplus S$ indicates that the scalar part is to be taken in the spaces of both spin A and spin B. Similarly, the probability that spin A, upon appropriate measurement, is found in state $\mathrm{P}_{\mathbf{s}_A}$ and spin B is found in state $\mathrm{P}_{\mathbf{s}_B}$ is $2^2 \left\langle \varrho_{AB} \mathrm{P}_{\mathbf{s}_A} \otimes \mathrm{P}_{\mathbf{s}_B} \right\rangle_{S \otimes S}$.

One way that linear superpositions of product states arise is in reduction of representations of the rotation group. If we rotate Ψ_{AB} by an additional rotation with rotor R we get

$$\Psi_{AB} \rightarrow R \otimes R \Psi_{AB}$$

and the rotation factor $R \otimes R$ is a product representation of $SU(2)$. For example, suppose we rotate each spin by π about the axis $\hat{\mathbf{a}}$. Then $R = \exp(-i\hat{\mathbf{a}}\pi/2) = -i\hat{\mathbf{a}}$ and $R \otimes R = -\hat{\mathbf{a}} \otimes \hat{\mathbf{a}}$.

The rotation factor $R \otimes R$ can be reduced to a direct sum of spin-1 and spin-0 parts:

$$R \otimes R \simeq UR \otimes RU^\dagger = D^{(0)} \oplus D^{(1)},$$

where the elements of the 4×4 unitary matrix U are Clebsch-Gordan coefficients $\langle SM|s_1 m_1 s_2 m_2\rangle$. The reduction entangles states that contribute to more than one irreducible representation. In particular, the $S, M = 1, 0$ and $S, M = 0, 0$ states are fully entangled. Consider the spin-0 part:

$$\psi^{(0)}_{AB} = \frac{1}{\sqrt{2}}(1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes 1)(\mathrm{P}_3 \otimes \mathrm{P}_3)$$

with a spin density

$$\varrho^{(0)}_{AB} = \psi^{(0)}_{AB}\psi^{(0)\dagger}_{AB} = \frac{1}{2}(1 - \mathbf{e}_1 \otimes \mathbf{e}_1)\left(\bar{\mathrm{P}}_3 \otimes \mathrm{P}_3 + \mathrm{P}_3 \otimes \bar{\mathrm{P}}_3\right).$$

Note that $\psi^{(0)}_{AB}$ for the singlet is antisymmetric under interchange, and $\varrho^{(0)}_{AB}$ is symmetric. One can also verify that $\psi^{(0)}_{AB}$ is invariant under rotations: $R \otimes R\psi^{(0)}_{AB} = \psi^{(0)}_{AB}$. It is easy to see that $\varrho^{(0)}_{AB}$ is idempotent and may be viewed as a 2-spin (or 2-qubit) projector. Thus, $\varrho^{(0)}_{AB}$ represents a pure state. Fully entangled states are pure. However, mixed states can also be partially entangled.

Consider the triplet state with $M_S = 0$. This is easily found within a normalization factor by starting with the $(S, M) = (1, 1)$ state

$$\psi^{(1,1)} = \mathrm{P}_3 \otimes \mathrm{P}_3$$

and applying the lowering operator [compare (49)] $S_- = \mathbf{e}_1\mathrm{P}_3 \otimes 1 + 1 \otimes \mathbf{e}_1\mathrm{P}_3$:

$$\psi^{(1,0)}_{AB} = \frac{1}{\sqrt{2}}(1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes 1)(\mathrm{P}_3 \otimes \mathrm{P}_3)$$

$$\varrho^{(1,0)}_{AB} = \psi^{(1,0)}_{AB}\psi^{(1,0)\dagger}_{AB} = \frac{1}{2}(1 + \mathbf{e}_1 \otimes \mathbf{e}_1)\left(\bar{\mathrm{P}}_3 \otimes \mathrm{P}_3 + \mathrm{P}_3 \otimes \bar{\mathrm{P}}_3\right).$$

Note $\frac{1}{2}(1 + \mathbf{e}_1 \otimes \mathbf{e}_1)$ and $\frac{1}{2}(1 - \mathbf{e}_1 \otimes \mathbf{e}_1)$ are complementary 2-spin projectors.

A mixture of equal amounts of fully entangled (and hence nonseparable) singlet and triplet states with $M = 0$ gives the state

$$\varrho' = \frac{1}{2}\left[\varrho^{(0)}_{AB} + \varrho^{(1,0)}_{AB}\right] = \frac{1}{2}\left(\bar{\mathrm{P}}_3 \otimes \mathrm{P}_3 + \mathrm{P}_3 \otimes \bar{\mathrm{P}}_3\right). \tag{50}$$

This is a *separable* state, that is a mixture of decomposable (uncorrelated) states. Through the mixture, the two spins do become correlated, but not entangled. The correlation is purely classical.

Total Spin and Nonlocal Projectors

However, we can project an entangled state out of the separable one (50) by applying a nonlocal projector. Consider the singlet-state projector for the two-spin system:

$$\mathrm{P}_{S=0} = 1 - \frac{1}{2}S^2,$$

where

$$S^2 = \sum_j S_j^2 = \frac{1}{4}\sum_{j=1}^{3} (\mathbf{e}_j \otimes 1 + 1 \otimes \mathbf{e}_j)^2 = \frac{1}{2}\sum_j (1 + \mathbf{e}_j \otimes \mathbf{e}_j)$$

is the square of the total spin operator. Substitution of S^2 into $\mathrm{P}_{S=0}$ gives directly the isotropic form

$$\begin{aligned}\mathrm{P}_{S=0} &= 1 - \frac{1}{4}\left[3 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\right] \\ &= \frac{1}{4}\left[1 - (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)\right] \ .\end{aligned}$$

Applying $\mathrm{P}_{S=0}$ to ϱ' (50) and noting that, with applications of the pacwoman property,

$$\mathrm{P}_{S=0}\left(\mathrm{P}_3 \otimes \bar{\mathrm{P}}_3 + \bar{\mathrm{P}}_3 \otimes \mathrm{P}_3\right) = \frac{1}{2}\left(1 - \mathbf{e}_1 \otimes \mathbf{e}_1\right)\left(\mathrm{P}_3 \otimes \bar{\mathrm{P}}_3 + \bar{\mathrm{P}}_3 \otimes \mathrm{P}_3\right)$$

we find

$$\begin{aligned}\mathrm{P}_{S=0}\varrho'\mathrm{P}_{S=0} &= \frac{1}{8}\left[1 - \mathbf{e}_1 \otimes \mathbf{e}_1\right]\left(\mathrm{P}_3 \otimes \bar{\mathrm{P}}_3 + \bar{\mathrm{P}}_3 \otimes \mathrm{P}_3\right)\left[1 - \mathbf{e}_1 \otimes \mathbf{e}_1\right] \\ &= \frac{1}{4}\left[1 - \mathbf{e}_1 \otimes \mathbf{e}_1\right]\left(\mathrm{P}_3 \otimes \bar{\mathrm{P}}_3 + \bar{\mathrm{P}}_3 \otimes \mathrm{P}_3\right),\end{aligned}$$

which is just $\frac{1}{2}\varrho_{AB}^{(0)}$, the singlet contribution to ϱ' (50). Note also that

$$\mathrm{P}_{S=1} = \frac{1}{2}S^2 = 1 - \mathrm{P}_{S=0}$$

is the projector onto the triplet state, and

$$\mathrm{P}_{S=1}\varrho'\mathrm{P}_{S=1} = \frac{1}{2}\varrho_{AB}^{(1,0)}.$$

The spin projectors $\mathrm{P}_{S=0}$ and $\mathrm{P}_{S=1}$ are examples of nonlocal projectors that can project unentangled states into entangled ones.

8. Conclusions

We have derived many quantum results for a classical system. The quantum/classical interface has become almost transparent in the spinor approach of Clifford's geometric algebra. Quantum effects demand *amplitudes*, and these are provided classically by rotors of the geometric algebra. In particular, the *eigenspinor* is an amplitude closely associated with quantum spinor wave functions. Linear equations for the classical eigenspinor suggest superposition and hence the possibility of quantum-like interference. Projectors onto minimal left ideals of the algebra are needed to reduce the spinor representation of Lorentz rotations in the algebra and to simplify the equation of motion, and by using them, many of the classical results take quantum form. The association helps clarify various quantum phenomena, and extensions to multiparticle systems promise to demystify such quantum phenomena as entanglement. The Q/C relation appears to be much deeper than superficial. The basic spinor representation gives spin 1/2 and changes in sign for rotations of 360°, and the linearity of the equation of motion gives $g = 2$.

While our results are concordant with most of quantum theory, there are some important differences in *interpretation*. In particular, fermion spin is given a classical foundation. Although quantum texts often give the magnitude of the spin as $\sqrt{3}\hbar/2$, there is no indication of any length greater than $\hbar/2$ in the classical approach. Since the projection of the spin in any direction is $\hbar/2$, a larger magnitude would precess about the quantization axis. Such a precession should have observable effects, but none has evidently been seen. The classical approach indicates that the prediction of a larger size is erroneous, arising from a misunderstanding of the spin component operators in an ideal spinor representation. A related statement from conventional quantum theory is that only one component of the spin can have a definite value at a time. This is usually justified by the uncertainty relation, which predicts uncertainties in the x and y components of the spin in any eigenstate of the z component. The classical approach suggests that in such an eigenstate, the the x and y components vanish and are therefore definite. However, any such state is a linear combination of spin up and down along $\mathbf{e}_1$ and any *measurement* of the x component forces the system into one of these states. The uncertainty relation is therefore correct but its interpretation questionable. The classical interpretation is that a spin direction $\mathbf{s}$ exists for each pure state, and this is reinforced by the spin density operator approach in quantum theory. The spin density operator for a pure quantum state is $\varrho = \psi\psi^\dagger = \frac{1}{2}(1 + \mathbf{s})$,which indicates a spin direction $\mathbf{s}$ identical to $R\mathbf{e}_3R^\dagger$ in the classical approach. Although ϱ cannot be determined by

a single measurement, it can be found by a set of measurements on different samples of a pure-state ensemble, in complete analogy with the process of measuring the Stokes' parameters in a fully polarized beam of light. The mathematics is in fact identical to the treatment of polarized light and our interpretation should probably be analogous.

The quantum age has started and quantum technology is on its way. It's time to shift gears into the paradigm of relativistic quantum theory, on which the quantum age is based. APS formulates mechanics in a fashion that naturally incorporates relativity and much of the structure of quantum theory. It seems destined to play a central role in the revision of undergraduate curriculum needed to complete the paradigm shift.

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