

Applications of Clifford Algebras in Physics

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ABSTRACT Clifford's geometric algebra is a powerful language for physics that clearly describes the geometric symmetries of both physical space and spacetime. Some of the power of the algebra arises from its natural spinorial formulation of rotations and Lorentz transformations in classical physics. This formulation brings important quantum-like tools to classical physics and helps break down the classical/quantum interface. It also unites Newtonian mechanics, relativity, quantum theory, and other areas of physics in a single formalism and language. This lecture is an introduction and sampling of a few of the important applications in physics.

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1 Introduction

Clifford's geometric algebra is an ideal language for physics because it maximally exploits geometric properties and symmetries. It is known to physicists mainly as the algebras of Pauli spin matrices and of Dirac gamma matrices, but its utility goes far beyond the applications to quantum theory and spin for which these matrix forms were introduced. In particular, Clifford algebra

- endows cross products with transparent geometric meanings,
- generalizes cross products to $n > 3$ dimensions, in particular to relativistic spacetime,
- clears potential confusion of pseudovectors and pseudoscalars,
- constructs the unit imaginary i as a geometric object, thereby explaining its important role in physics and extending complex analysis to more than two dimensions,

- reduces rotations and Lorentz transformations to algebraic multiplication, and more generally
- allows computational geometry without matrices or tensors,
- formulates classical physics in an efficient spinorial formulation with tools that are closely related to ones familiar in quantum theory such as spinors (“rotors”) and projectors, and
- thereby unites Newtonian mechanics, relativity, quantum theory, and more in a single formalism and language that is as simple as the algebra of the Pauli spin matrices.

In this lecture, there is only enough space to discuss a few representative high points of the many applications of Clifford algebras in physics. For other applications, the reader will be referred to published articles and books. The mathematical background for this chapter is given by the lecture[1] of the late Professor Pertti Lounesto earlier in the volume.

2 Three Clifford Algebras

The three most commonly employed Clifford algebras in physics are the quaternion algebra $\mathbb{H} = \mathcal{C}\ell_{0,2}$, the algebra of physical space (APS) $\mathcal{C}\ell_3$, and the spacetime algebra (STA) $\mathcal{C}\ell_{1,3}$. They are closely related. Hamilton’s quaternion algebra,[2] introduced in 1843 to handle rotations, is the oldest, and provided the source of the concept (again, by Hamilton) of vectors. The superiority of \mathbb{H} for matrix-free and coordinate-free computations of rotations has been recently rediscovered by space programs, the computer-games industry, and robotics engineering. Furthermore, \mathbb{H} is a division algebra and has been investigated as a replacement of the complex field in an extension of Dirac theory.[3] Quaternions were used by Maxwell and Tate to express Maxwell’s equations of electromagnetism in compact form, and they motivated Clifford to find generalizations based on Grassmann theory. Hamilton’s biquaternions (complex quaternions) are quaternions over the complex field $\mathbb{H} \times \mathbb{C}$, and with them, Conway (1911) and others were able to express Maxwell’s equations as a single equation.

The complex quaternions are isomorphic to APS: $\mathbb{H} \times \mathbb{C} \simeq \mathcal{C}\ell_3$, which is familiar to physicists as the algebra of the Pauli spin matrices. The even subalgebra $\mathcal{C}\ell_3^+$ is isomorphic to \mathbb{H} , and the correspondences $\mathbf{i} \leftrightarrow \mathbf{e}_{32}$, $\mathbf{j} \leftrightarrow \mathbf{e}_{13}$, $\mathbf{k} \leftrightarrow \mathbf{e}_{21}$ identify pure quaternions with bivectors in APS, and hence with generators of rotations in physical space. APS distinguishes cleanly between vectors and bivectors, in contrast to most approaches with complex quaternions. The volume element \mathbf{e}_{123} in APS can be identified with the unit imaginary i since it squares to -1 and commutes with vectors and their products. Every element of APS can then be expressed as

a complex *paravector*, that is the sum of a scalar and a vector.[4, 5, 6] The identification $i = \mathbf{e}_{123}$ endows the unit imaginary with geometrical significance and helps explain the widespread use of complex numbers in physics.[7] The sign of i is reversed under parity inversion, and imaginary scalars and vectors correspond to pseudoscalars and pseudovectors, respectively. The dual of any element x is simply $-ix$, and in particular, one sees that the vector cross product $\mathbf{x} \times \mathbf{y}$ of two vectors \mathbf{x}, \mathbf{y} is the vector dual to the bivector $\mathbf{x} \wedge \mathbf{y}$. It is traditional in APS to denote reversal by a dagger, $\hat{x} = x^\dagger$, since reversal corresponds to Hermitian conjugation in any matrix representation that uses Hermitian matrices (such as the Pauli spin matrices) for the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The quadratic form of vectors in APS is the traditional dot product and implies a Euclidean metric. Paravectors constitute a four-dimensional linear subspace of APS, but as shown below, the quadratic form they inherit implies the metric of Minkowski spacetime rather than Euclidean space. Paravectors can therefore be used to model spacetime vectors in relativity.[8, 9] The algebra of paravectors and their products is no different from the algebra of vectors, and in particular the paravector volume element is i and duals are defined in the same way. APS admits interpretations in both spacetime (paravector) and spatial (vector) terms, and the formulation of relativity in APS marries covariant spacetime notation with the vector notation of spatial vectors. Matrix elements and tensor components are not needed, although they can be obtained by expanding multiparavectors in the basis elements of APS.

Another approach to spacetime is to introduce the Minkowski metric directly in $\mathcal{C}\ell_{1,3}$ or $\mathcal{C}\ell_{3,1}$. Thus, $\mathcal{C}\ell_{1,3}$ is generated by products of the basis vectors γ_μ , $\mu = 0, 1, 2, 3$, satisfying

$$\frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0 \\ -1, & \mu = \nu = 1, 2, 3 \\ 0, & \mu \neq \nu \end{cases} .$$

Dirac's electron theory (1928) was based on a matrix representation of $\mathcal{C}\ell_{1,3}$ over the complex field, and Hestenes[10] (1966) pioneered the use of STA (real $\mathcal{C}\ell_{1,3}$) in several areas of physics. The even subalgebra is isomorphic to APS: $\mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_3$. The volume element in STA is $\mathbf{I} = \gamma_0\gamma_1\gamma_2\gamma_3$. Although it is referred to as the unit pseudoscalar and squares to -1 , it anticommutes with vectors, thus behaving more like an additional spatial dimension than a scalar.

This lecture mainly uses APS, although generalizations to $\mathcal{C}\ell_n$ are made where convenient, and one section is devoted to the relation of APS to STA.

2.1 Bivectors as Plane Areas

The Clifford (or geometric) product of vectors was introduced in Lecture 1 and given there as the sum of dot and wedge products. The dot product is familiar from traditional vector analysis, but the wedge product of two vectors is a new entity: the bivector. The bivector $\mathbf{v} \wedge \mathbf{w}$ represents the plane containing \mathbf{v} and \mathbf{w} ; as discussed in Lecture 1, it has an intrinsic orientation given by the circulation in the plane from \mathbf{v} to \mathbf{w} and a size given by the area of the parallelogram with sides \mathbf{v} and \mathbf{w} . Bivectors enter physics in many ways: as areas, planes, and the generators of reflections and rotations. In the old vector analysis of Gibbs and Heaviside, bivectors are replaced by cross products that give vectors perpendicular to the plane. However, this ploy is only useful in three dimensions, and it hides the intrinsic properties of bivectors. As discussed below, cross products are actually examples of algebraic duals.

The conjugation (antiautomorphism, anti-involution) of *reversal*, as defined in Lecture 1, reverses the order of vector factors in the product. Because noncollinear vectors do not commute, this conjugation gives us a way of distinguishing collinear and orthogonal components. Any element invariant under reversal is said to be *real* whereas elements that change sign are *imaginary*. Every element x can be split into real and imaginary parts:

$$x = \frac{x + x^\dagger}{2} + \frac{x - x^\dagger}{2} \equiv \langle x \rangle_{\Re} + \langle x \rangle_{\Im} .$$

Scalars and vectors (in a Euclidean space) are thus real, whereas bivectors are imaginary.

Exercise 1 Verify that the dot and wedge product of any vectors $\mathbf{u}, \mathbf{v} \in \mathcal{C}_n$ can be identified as the real and imaginary parts of the geometric product $\mathbf{u}\mathbf{v}$.

Exercise 2 Consider the triangle of vectors $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Prove

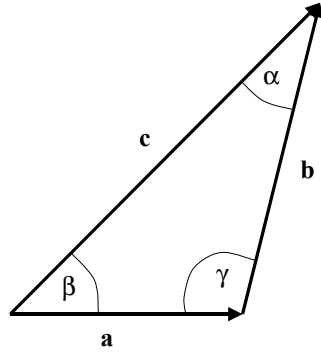
$$\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c}$$

and show that the magnitude of these wedge products is twice the area of the triangle.

Exercise 3 Let α, β, γ be the interior angles of the triangle (see last exercise) opposite sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Use the relation of the wedge products in the previous exercise to prove the law of sines:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c},$$

where a, b, c are the magnitudes of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

FIGURE 1. A triangle of vectors $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

Exercise 4 Let \mathbf{r} be the position vector of a point that moves with velocity $\mathbf{v} = \dot{\mathbf{r}}$. Show that the magnitude of the bivector $\mathbf{r} \wedge \mathbf{v}$ is twice the rate at which the time-dependent \mathbf{r} sweeps out area. Explain how this relates the conservation of angular momentum $\mathbf{r} \wedge \mathbf{p}$, with $\mathbf{p} = m\mathbf{v}$, to Kepler's second law for planetary orbits, namely that equal areas are swept out in equal times.

2.2 Bivectors as operators

The fact that the bivector of a plane commutes with vectors orthogonal to the plane and anticommutes with ones in the plane means that we can easily use unit bivectors to represent reflections. In particular, in an n -dimensional Euclidean space, $n > 2$, the two-sided transformation

$$\mathbf{v} \rightarrow \mathbf{e}_{12}\mathbf{v}\mathbf{e}_{12}$$

reflects any vector \mathbf{v} in the $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$ plane, as is verified in the next exercise.

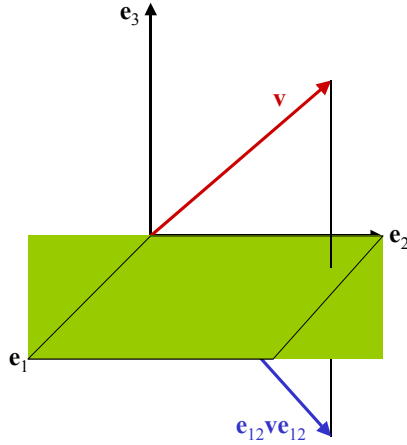
Exercise 5 Expand $\mathbf{v} = v^k\mathbf{e}_k$ in the n -dimensional basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to prove

$$\mathbf{e}_{12}\mathbf{v}\mathbf{e}_{12} = 2(v^1\mathbf{e}_1 + v^2\mathbf{e}_2) - \mathbf{v} = \mathbf{v}^\Delta - \mathbf{v}^\perp,$$

where \mathbf{v}^Δ is the component of \mathbf{v} coplanar with \mathbf{e}_{12} and $\mathbf{v}^\perp = \mathbf{v} - \mathbf{v}^\Delta$ is the component orthogonal to the plane. In words, components in the \mathbf{e}_{12} plane remain unchanged, but those orthogonal to the plane change sign. This is what we mean by a reflection in the \mathbf{e}_{12} plane.

Exercise 6 Show that the coplanar component of \mathbf{v} is given by

$$\mathbf{v}^\Delta = \frac{1}{2}(\mathbf{v} + \mathbf{e}_{12}\mathbf{v}\mathbf{e}_{12}).$$

FIGURE 2. The reflection of \mathbf{v} in the plane \mathbf{e}_{12} is $\mathbf{e}_{12}\mathbf{v}\mathbf{e}_{12}$.

Find a similar expression for the orthogonal component \mathbf{v}^\perp .

While the representation of reflections by an algebraic product is useful in many physical applications, the fact that bivectors generate rotations is of fundamental importance in physics. We therefore devote part of this lecture to expanding the basic formulation of rotations in $\mathcal{C}\ell_2$ introduced in Lecture 1. Try the following:

Exercise 7 Simplify the products $\mathbf{e}_1\mathbf{e}_{12}$ and $\mathbf{e}_2\mathbf{e}_{12}$.

Note that both \mathbf{e}_1 and \mathbf{e}_2 are rotated in the same direction through 90 degrees by right-multiplication with \mathbf{e}_{12} . The same result is given by left-multiplication with $\mathbf{e}_{21} = \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_{12}^\dagger = -\mathbf{e}_{12}$:

$$\begin{aligned}\mathbf{e}_1\mathbf{e}_{12} &= \mathbf{e}_2 = \mathbf{e}_{21}\mathbf{e}_1 \\ \mathbf{e}_2\mathbf{e}_{12} &= -\mathbf{e}_1 = \mathbf{e}_{21}\mathbf{e}_2\end{aligned}$$

It follows that *any* vector $\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$ in the \mathbf{e}_{12} plane is rotated by 90° when multiplied by the unit bivector \mathbf{e}_{12} : $\mathbf{v} \rightarrow \mathbf{v}\mathbf{e}_{12}$ (see Fig. 3).

Exercise 8 Find the operator that upon multiplication from the right rotates any vector in the \mathbf{e}_{12} plane by the small angle $\alpha \ll 1$. This should be expressed as a first-order approximation in α .

To rotate by an angle θ other than 90°, we take a linear combination of 1 and \mathbf{e}_{12} to form the rotation operator

$$\cos\theta + \mathbf{e}_{12}\sin\theta = \exp(\mathbf{e}_{12}\theta). \quad (2.1)$$

The *Euler relation* for the bivector follows from that for complex numbers: it depends only on $(\mathbf{e}_{12})^2 = -1$. The bivector \mathbf{e}_{12} thus *generates a rotation*

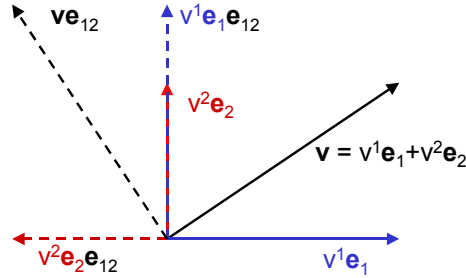


FIGURE 3. The bivector \mathbf{e}_{12} rotates vectors in the \mathbf{e}_{12} plane by 90° .

in the \mathbf{e}_{12} plane: any vector \mathbf{v} in the \mathbf{e}_{12} plane is rotated by θ in the plane in the sense that takes $\mathbf{e}_1 \rightarrow \mathbf{e}_2$ by

$$\mathbf{v} \rightarrow \mathbf{v} \exp(\mathbf{e}_{12}\theta) = \exp(\mathbf{e}_{21}\theta) \mathbf{v} . \quad (2.2)$$

Exercise 9 Verify that the operator $\exp(\mathbf{e}_{12}\theta)$ has the appropriate limit $\theta \rightarrow \pi/2$ and when $0 < \theta \ll 1$ (see previous exercise).

A finite rotation is performed smoothly by increasing θ gradually from 0 to its full value. To represent a continual rotation in the \mathbf{e}_{12} plane at the angular rate ω , we put $\theta = \omega t$. Note that the rotation element $\exp(\mathbf{e}_{12}\theta)$ is the product of two unit vectors in the \mathbf{e}_{12} plane:

$$\exp(\mathbf{e}_{12}\theta) = \mathbf{e}_1 \mathbf{e}_1 \exp(\mathbf{e}_{12}\theta) \equiv \mathbf{e}_1 \mathbf{n} ,$$

where $\mathbf{n} = \mathbf{e}_1 \exp(\mathbf{e}_{12}\theta)$ is the unit vector obtained from \mathbf{e}_1 by a rotation of θ in the \mathbf{e}_{12} plane.

Exercise 10 Expand $\mathbf{n} = \mathbf{e}_1 \exp(\mathbf{e}_{12}\theta)$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and verify that $\mathbf{e}_1 \mathbf{n} = \cos \theta + \mathbf{e}_{12} \sin \theta$. Show that the scalar and bivector parts of $\exp(\mathbf{e}_{12}\theta)$ are equal to $\mathbf{e}_1 \cdot \mathbf{n}$ and $\mathbf{e}_1 \wedge \mathbf{n}$, respectively.

In general, every product $\mathbf{m} \mathbf{n}$ of unit vectors \mathbf{m} and \mathbf{n} can be interpreted as a rotation operator of the form $\exp(\hat{\mathbf{B}}\theta)$, where the unit bivector $\hat{\mathbf{B}}$ represents the plane containing \mathbf{m} and \mathbf{n} , and θ is the angle between them. The product $\mathbf{m} \mathbf{n}$ does not depend on the actual directions of \mathbf{m} and \mathbf{n} , but only on the plane in which they lie and on the angle between them.

Exercise 11 Let $\mathbf{a} = \beta \mathbf{e}_1 \exp(\mathbf{e}_{12}\theta)$ and $\mathbf{b} = \beta^{-1} \mathbf{e}_2 \exp(\mathbf{e}_{12}\theta)$ be vectors obtained by rotating \mathbf{e}_1 and \mathbf{e}_2 through the angle θ in the \mathbf{e}_{12} plane and then dilating by complimentary factors. Prove that $\mathbf{a} \mathbf{b} = \mathbf{e}_{12}$. [Hint: note that \mathbf{b} can also be written $\beta^{-1} \exp(-\mathbf{e}_{12}\theta) \mathbf{e}_2$ and that $\exp(\mathbf{e}_{12}\theta) \exp(-\mathbf{e}_{12}\theta) = 1$.]

The last exercise shows that the product \mathbf{ab} of two perpendicular vectors depends only on the area and orientation of the rectangle they form. The product does *not* depend on the individual directions and lengths of the vectors. More generally, the wedge product $\mathbf{a} \wedge \mathbf{b}$ of two vectors \mathbf{a}, \mathbf{b} depends only on the area and orientation of the parallelogram they form.

Rotations in Spaces of More Than Two Dimensions

Relation (2.2) shows two ways to rotate any vector \mathbf{v} in a plane. Note that the left and right rotation operators are *not* the same, but rather inverses of each other: $\exp(\mathbf{e}_{21}\theta)\exp(\mathbf{e}_{12}\theta) = \exp(\mathbf{e}_{21}\theta)\exp(-\mathbf{e}_{21}\theta) = 1$.

Exercise 12 Use the Euler relation to expand the exponentials $\exp(\mathbf{B}_1)$ and $\exp(\mathbf{B}_2)$ of bivectors $\mathbf{B}_1 = \theta_1 \hat{\mathbf{B}}_1$ and $\mathbf{B}_2 = \theta_2 \hat{\mathbf{B}}_2$, where $\hat{\mathbf{B}}_1^2 = \hat{\mathbf{B}}_2^2 = -1$. Prove that if $\hat{\mathbf{B}}_1 = \pm \hat{\mathbf{B}}_2$, then

$$\exp(\mathbf{B}_1)\exp(\mathbf{B}_2) = \exp(\mathbf{B}_1 + \mathbf{B}_2) .$$

Also show that when $\mathbf{B}_1\mathbf{B}_2 \neq \mathbf{B}_2\mathbf{B}_1$, the relation $\exp(\mathbf{B}_1)\exp(\mathbf{B}_2) = \exp(\mathbf{B}_1 + \mathbf{B}_2)$ is not generally valid.

In spaces of more than two dimensions, we may want to rotate a vector \mathbf{v} that has components perpendicular to the rotation plane. Then the products $\mathbf{v}\exp(\mathbf{e}_1\mathbf{e}_2\theta)$ and $\exp(\mathbf{e}_2\mathbf{e}_1\theta)\mathbf{v}$ no longer work; they include terms like $\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2$ that are not even vectors. (They are trivectors as will be discussed in the next section.) We need a form for the rotation that leaves perpendicular components invariant. Recall that perpendicular vectors commute with the bivector for the plane. For example, $\mathbf{e}_3\mathbf{e}_{12} = \mathbf{e}_{12}\mathbf{e}_3$. We therefore use

$$\mathbf{v} \rightarrow R\mathbf{v}R^{-1} \tag{2.3}$$

for the rotation, where $R = \exp(\mathbf{e}_{21}\theta/2)$ is a *rotor* and $R^{-1} = \exp(\mathbf{e}_{12}\theta/2)$ is its inverse. The transformation (2.3) is linear, because $R\alpha\mathbf{v}R^{-1} = \alpha R\mathbf{v}R^{-1}$ for any scalar α , and $R(\mathbf{v} + \mathbf{w})R^{-1} = R\mathbf{v}R^{-1} + R\mathbf{w}R^{-1}$ for any two vectors \mathbf{v} and \mathbf{w} . The two-sided form of the rotation leaves anything that commutes with the rotation plane invariant. This includes vector components perpendicular to the rotation plane as well as scalars. The form may remind you of transformations of operators in quantum theory. It is sometimes called a *spin transformation* to distinguish it from one-sided transformations common with matrices operating on column vectors.

Exercise 13 Show that rotors in Euclidean spaces are unitary: $R^{-1} = R^\dagger$.

The ability to rotate in any plane of n -dimensional space without components, tensors, or matrices is a major strength of geometric algebra in physics. A product of rotors is a rotor (rotations form a group), and in

physical space, where $n = 3$, any rotor can be factored into a product of Euler-angle rotors

$$R = \exp\left(-\mathbf{e}_{12}\frac{\phi}{2}\right) \exp\left(-\mathbf{e}_{31}\frac{\theta}{2}\right) \exp\left(-\mathbf{e}_{12}\frac{\psi}{2}\right). \quad (2.4)$$

However, such factorization is not necessary since there is always a simpler expression $R = \exp(\Theta)$, where Θ is a bivector whose orientation gives the plane of rotation and whose magnitude gives half the angle of rotation. By using the rotor $R = \exp(\Theta)$ instead of a product of three rotations about specified angles, one can avoid degeneracies in the Euler-angle decomposition. The simple rotor expression allows smooth rotations in a single plane and thus interpolations between arbitrary orientations.

Exercise 14 Show that the magnitude of Θ in the rotor $R = \exp(\Theta)$ is the area swept out by any unit vector \mathbf{n} in the rotation plane under the rotation $\mathbf{n} \rightarrow R\mathbf{n}R^{-1}$. [Hint: Note that the increment in area added when the unit vector is rotated through the incremental angle $d\theta$ is $\frac{1}{2}d\theta$.]

Exercise 15 Consider the Euler-angle rotor (2.4). Show that when $\theta = 0$ the result depends only on $\phi + \psi$ and is independent of the value $\phi - \psi$, whereas when $\theta = \pi$, the converse holds. Such degeneracies can cause problems if Euler angles are used in robotic or 3-D video applications.

Relation to Rotation by Matrix Multiplication.

Spin transformations rotate vectors, and when we expand $\mathbf{v} = v^j \mathbf{e}_j$, it is the basis vectors and not the coefficients that are directly rotated:

$$\begin{aligned} \mathbf{v} &= v^j \mathbf{e}_j \rightarrow \mathbf{v}' = v^j \mathbf{e}'_j \\ \mathbf{e}_j &\rightarrow \mathbf{e}'_j = R\mathbf{e}_jR^{-1}. \end{aligned}$$

It is easy to find the components of the rotated vector \mathbf{v}' in the original (unprimed) basis:

$$v'^i = \mathbf{e}_i \cdot \mathbf{v}' = (\mathbf{e}_i \cdot \mathbf{e}'_j) v^j.$$

The matrix of values $(\mathbf{e}_i \cdot \mathbf{e}'_j)$ is the usual rotation matrix.¹ For example, with $R = \exp(\mathbf{e}_{21}\theta/2)$,

$$\begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

¹While one can readily compute the transformation matrices by writing the spin transformations explicitly for basis vectors, neither the matrices nor even the vector components are needed in the algebraic formulation.

Exercise 16 Show that if $\mathbf{n} = R\mathbf{e}_1R^\dagger = \exp(\mathbf{e}_{21}\theta)\mathbf{e}_1$, where $R = \exp(\mathbf{e}_{21}\theta/2)$, then

$$R = (\mathbf{n}\mathbf{e}_1)^{1/2} = \frac{(\mathbf{n}\mathbf{e}_1 + 1)}{\sqrt{2(1 + \mathbf{n} \cdot \mathbf{e}_1)}}.$$

[Hint: find the unit vector that bisects \mathbf{n} and \mathbf{e}_1 .]

Relation of Rotations to Reflections

Evidently $R^2 = \exp(\mathbf{e}_{21}\theta) = \mathbf{n}\mathbf{e}_1$ is the rotor for a rotation in the plane of \mathbf{n} and \mathbf{e}_1 by 2θ . Its inverse is $\mathbf{e}_1\mathbf{n}$, and it takes any vector \mathbf{v} into

$$\mathbf{v} \rightarrow \mathbf{n}\mathbf{e}_1\mathbf{v}\mathbf{e}_1\mathbf{n}.$$

If \mathbf{e}_3 is a unit vector normal to the plane of rotation,

$$\begin{aligned} \mathbf{v} &\rightarrow \mathbf{n}\mathbf{e}_3\mathbf{e}_1\mathbf{v}\mathbf{e}_3\mathbf{e}_1\mathbf{n}\mathbf{e}_3 \\ &= (\mathbf{n}\mathbf{e}_3)\mathbf{e}_{31}\mathbf{v}\mathbf{e}_{31}(\mathbf{n}\mathbf{e}_3), \end{aligned}$$

which represents the rotation as reflections in two planes with unit bivectors $\mathbf{n}\mathbf{e}_3$ and \mathbf{e}_{31} . The planes intersect along \mathbf{e}_3 and have a dihedral angle of θ . See Fig. (4).

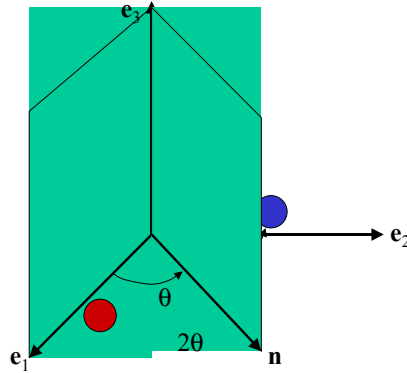


FIGURE 4. Successive reflections in two planes is equivalent to a rotation by twice the dihedral angle θ of the planes. In the diagram, the red ball is first reflected in the $\mathbf{e}_3\mathbf{e}_1$ plane and then in the $\mathbf{n}\mathbf{e}_3$ plane.

Example 17 Mirrors in clothing stores are often arranged to give double reflections so that you can see yourself rotated rather than reflected. Two mirrors with a dihedral angle of 90° will rotate your image by 180° . This corresponds to the above transformation with \mathbf{n} replaced by \mathbf{e}_2 .

Exercise 18 How could you orient two mirrors so that you see yourself from the side, that is, rotated by 270° ?

What if we rotate the reflection planes in the \mathbf{e}_{12} plane? (In physical space, we might speak of rotating the planes about the \mathbf{e}_3 axis, but of course this makes sense only in three dimensions.) Rotating the plane means rotating its bivector, and to rotate \mathbf{ne}_3 , for example, we rotate both \mathbf{n} and \mathbf{e}_3 (although here, \mathbf{e}_3 is invariant under rotations in \mathbf{e}_{12}):

$$\mathbf{ne}_3 \rightarrow R\mathbf{n}R^{-1}R\mathbf{e}_3R^{-1} = R\mathbf{ne}_3R^{-1} = \mathbf{ne}_3R^{-2}.$$

Similarly,

$$\mathbf{e}_{31} \rightarrow R^2\mathbf{e}_{31}.$$

The product $(\mathbf{ne}_3)\mathbf{e}_{31}$ is invariant. We conclude that the rotation that results from successive reflections in two nonparallel planes in physical space depends only on the line of intersection and the dihedral angle between the planes; it is independent of rotations for both planes about their common axis.

Exercise 19 *Corner cubes are used on the moon and in the rear lenses on cars to reverse the direction of the incident light. Consider a sequence of three reflections, first in the \mathbf{e}_{12} plane, followed by one in the \mathbf{e}_{23} plane, followed by one in the \mathbf{e}_{31} plane. Show that when applied to any vector \mathbf{v} , the result is $-\mathbf{v}$.*

Spatial Rotations as Spherical Vectors

Any rotation is specified by the plane of rotation and the area swept out by a unit vector in the plane under the rotation. In physical space, as the rotation proceeds, the unit vector sweeps out a path on the surface S^2 of a unit sphere; this path can represent the rotation. The rotation plane includes the origin and intersects S^2 in a great circle, and the path representing the rotation is a *directed arc* on this great circle. We call such a directed arc a *spherical vector*. Spherical vectors are as straight as they can be on S^2 , and they can be freely translated along their great circles. However, spherical vectors on different great circles represent rotations on different planes and are generally distinct.

We have seen that any rotor $R = \exp \Theta$ is the product of two unit vectors \mathbf{a}, \mathbf{b} ,

$$R = \exp \Theta = \mathbf{ba},$$

where \mathbf{a} and \mathbf{b} lie in the plane of Θ and the angle between them is Θ . (Points on S^2 are the ends of unit vectors from the origin of the sphere, and we represent both with the same bold-face symbol.) The spherical vector $\overrightarrow{\mathbf{ab}}$ on S^2 from \mathbf{a} to \mathbf{b} represents R .

Exercise 20 *If $R = \mathbf{ba}$, then $R^\dagger = \mathbf{ab}$. Verify with these expressions that R and R^\dagger are inverses of each other.*

Exercise 21 Show that $\mathbf{ba} = (R\mathbf{b}R^{-1})(R\mathbf{a}R^{-1})$ for any rotor $R = \exp(\alpha\hat{\Theta})$ in the plane of \mathbf{a} and \mathbf{b} .

Now let's combine R with a rotor in a different plane, say R' . Distinct planes have distinct great circles on S^2 and intersect at antipodal points. We slide \mathbf{a} and \mathbf{b} along the great circle of Θ until \mathbf{b} is at one of the intersections. Then we choose \mathbf{c} so that $R' = \mathbf{cb}$. The composition

$$R'R = \mathbf{cb} \mathbf{ba} = \mathbf{ca}$$

yields a rotor \mathbf{ca} represented by the spherical vector $\vec{\mathbf{ac}} = \vec{\mathbf{ab}} + \vec{\mathbf{bc}}$ from \mathbf{a} to \mathbf{c} . The composition of rotations thus is equivalent to the *addition of spherical vectors* on S^2 (see Fig. 5).

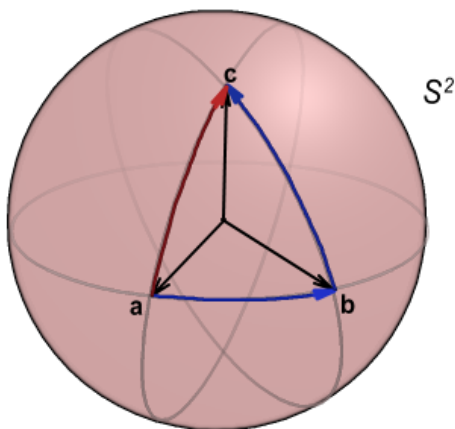


FIGURE 5. A product of rotations is represented by the addition of spherical vectors.

The length of the spherical vector $\vec{\mathbf{ab}}$ from \mathbf{a} to \mathbf{b} , which represents the rotor $R = \mathbf{ba}$, is *half* the maximum angle of rotation of a vector $\mathbf{v} \rightarrow R\mathbf{v}R^{-1}$. In other words, the *length* of $\vec{\mathbf{ab}}$ is the *area* swept out by a *unit* vector in the rotation plane. Points on S^2 are not directly associated with directions in physical space. *Pairs* of points on S^2 separated by an angle θ represent rotors in physical space that rotate vectors by up to an angle of 2θ ; the points themselves are associated with *spinors*. (The points do not fully identify the spinors but only their *poles*. Their orientation about the poles requires an additional complex phase which is not required for the treatment of rotations.)

We refer to S^2 as the *Cartan sphere*.² It is not to be confused with the unit sphere in physical space. Indeed, there is a two-to-one mapping of

²In recognition of Élie Cartan's extensive work with spinor representations of simple

points from S^2 to directions in physical space: antipodes on the Cartan sphere map to the same direction in physical space.³ Note that the addition of spherical vectors is noncommutative. This reflects the noncommutativity of rotations in different planes.

Example 22 *What's the product of a 180° rotation in the \mathbf{e}_{12} plane followed by a 180° rotation in the \mathbf{e}_{23} plane? Use the Euler relation $\exp(\mathbf{e}_{21}\alpha) = \cos\alpha + \mathbf{e}_{21}\sin\alpha$ to get*

$$\exp\left(\mathbf{e}_{32}\frac{\pi}{2}\right)\exp\left(\mathbf{e}_{21}\frac{\pi}{2}\right) = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3\mathbf{e}_1 = \exp\left(\mathbf{e}_{31}\frac{\pi}{2}\right).$$

The result is therefore a 180° rotation in the \mathbf{e}_{13} plane. Note that we do not need to compute an entire rotation $R\mathbf{v}R^{-1}$ but only the rotor R . In terms of spherical vectors, the composition is equivalent to adding a 90° vector on the equator to one joining the equator to the north pole.

Exercise 23 *Show that the result of a 90° rotation in the \mathbf{e}_{12} plane followed by a 90° rotation in the \mathbf{e}_{23} plane is a 120° rotation in the plane $(\mathbf{e}_{12} + \mathbf{e}_{23} + \mathbf{e}_{31})/\sqrt{3}$.*

We now have both an algebraic way and a geometric way to rotate any vector by any angle in any plane, and their relation provides simple algebraic calculations for spherical trigonometry. If $\hat{\mathbf{B}}$ is the unit bivector for the plane and θ is the angle of rotation, the vector \mathbf{v} under rotation becomes

$$\begin{aligned}\mathbf{v} &\rightarrow \mathbf{v}' = R\mathbf{v}R^{-1} \\ R &= \exp\left(\hat{\mathbf{B}}\theta/2\right).\end{aligned}$$

If, for example, $\hat{\mathbf{B}} = \mathbf{e}_{21} = \mathbf{e}_2\mathbf{e}_1$, the sense of the rotation is from \mathbf{e}_1 towards \mathbf{e}_2 . The rotation can be evaluated algebraically without components or matrices. For the algebraic calculation, one can expand $R = \cos\theta/2 + \hat{\mathbf{B}}\sin\theta/2$, but it is much easier to first expand \mathbf{v} into components in the plane of rotation (coplanar: Δ) and orthogonal (\perp) to it:

$$\mathbf{v} = \mathbf{v}^\Delta + \mathbf{v}^\perp.$$

Since \mathbf{v}^Δ anticommutes with $\hat{\mathbf{B}}$ whereas \mathbf{v}^\perp commutes with it,

$$\begin{aligned}R\mathbf{v}R^{-1} &= R^2\mathbf{v}^\Delta + \mathbf{v}^\perp \\ &= \mathbf{v}^\Delta \cos\theta + \hat{\mathbf{B}}\mathbf{v}^\Delta \sin\theta + \mathbf{v}^\perp.\end{aligned}$$

As before, the product $\hat{\mathbf{B}}\mathbf{v}^\Delta$ of a unit bivector with a vector in the plane of the bivector, rotates that vector by a right angle in the plane.

Lie algebras.

³Spherical vectors on S^2 give a faithful representation of rotations in $SU(2)$, the double covering group of $SO(3)$.

Exercise 24 Expand R^{-1} to prove that $\mathbf{v}^\Delta R^{-1} = R\mathbf{v}^\Delta$ and $\mathbf{v}^\perp R^{-1} = R^{-1}\mathbf{v}^\perp$, where \mathbf{v}^Δ lies in the plane of the rotation (is coplanar) and \mathbf{v}^\perp is orthogonal to the rotation plane.

Time-dependent Rotations

An additional infinitesimal rotation by $\boldsymbol{\Omega}'dt$ during the time interval dt changes a rotor R to

$$R + dR = \left(1 + \frac{1}{2}\boldsymbol{\Omega}'dt\right) R.$$

The time-rate of change of R thus has the form

$$\dot{R} = \frac{dR}{dt} = \frac{1}{2}\boldsymbol{\Omega}'R \equiv \frac{1}{2}R\boldsymbol{\Omega},$$

where the bivector $\boldsymbol{\Omega} = R^{-1}\boldsymbol{\Omega}'R$ is the *rotation rate* as viewed in the rotating frame. For the special case of a constant rotation rate, we can take the rotor to be

$$R(t) = R(0)e^{\boldsymbol{\Omega}t/2}.$$

If $R(0) = 1$, then $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$. Any vector \mathbf{r} is thereby rotated to

$$\mathbf{r}' = R\mathbf{r}R^{-1}$$

giving a time derivative

$$\begin{aligned}\dot{\mathbf{r}}' &= R \left[\frac{\boldsymbol{\Omega}\mathbf{r} - \mathbf{r}\boldsymbol{\Omega}}{2} + \dot{\mathbf{r}} \right] R^{-1} \\ &= R [\langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} + \dot{\mathbf{r}}] R^{-1},\end{aligned}$$

where we noted that \mathbf{r} is real and the bivector $\boldsymbol{\Omega}$ is imaginary. Since \mathbf{r} can be any vector, we can replace it by $\langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} + \dot{\mathbf{r}}$ to determine the second derivative

$$\begin{aligned}\ddot{\mathbf{r}}' &= R [\langle \boldsymbol{\Omega}(\langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} + \dot{\mathbf{r}}) \rangle_{\Re} + \langle \boldsymbol{\Omega}\dot{\mathbf{r}} \rangle_{\Re} + \ddot{\mathbf{r}}] R^{-1} \\ &= R [\langle \boldsymbol{\Omega} \langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} \rangle_{\Re} + 2 \langle \boldsymbol{\Omega}\dot{\mathbf{r}} \rangle_{\Re} + \ddot{\mathbf{r}}] R^{-1}.\end{aligned}\tag{2.5}$$

Let's write $\boldsymbol{\Omega} = \omega\hat{\boldsymbol{\Omega}}$. Then $\langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} = \boldsymbol{\Omega}\mathbf{r}^\Delta = \omega\hat{\boldsymbol{\Omega}}\mathbf{r}^\Delta$, where \mathbf{r}^Δ is the part of \mathbf{r} coplanar with $\hat{\boldsymbol{\Omega}}$. The product $\hat{\boldsymbol{\Omega}}\mathbf{r}^\Delta$ is \mathbf{r}^Δ rotated by a right angle in the plane $\hat{\boldsymbol{\Omega}}$.

Exercise 25 Show that $\langle \boldsymbol{\Omega} \langle \boldsymbol{\Omega}\mathbf{r} \rangle_{\Re} \rangle_{\Re} = -\omega^2\mathbf{r}^\Delta$ and that the minus sign can be viewed as arising from two 90-degree rotations or, equivalently, from the square of a unit bivector.

The result can be expressed

$$\ddot{\mathbf{r}}' = R \left[-\omega^2 \mathbf{r}^\Delta + 2\omega \hat{\mathbf{\Omega}} \dot{\mathbf{r}}^\Delta + \ddot{\mathbf{r}} \right] R^{-1}.$$

A force law $\mathbf{f}' = m\ddot{\mathbf{r}}'$ in the inertial system is seen to be equivalent to an effective force

$$\mathbf{f} = m\ddot{\mathbf{r}} = R^{-1}\mathbf{f}'R + m\omega^2 \mathbf{r}^\Delta + 2m\omega \dot{\mathbf{r}}^\Delta \hat{\mathbf{\Omega}}$$

in the rotating frame. The second and third terms on the RHS are identified as the centrifugal and Coriolis forces, respectively.

2.3 Higher-Grade Multivectors

Higher-order products of vectors also play important roles in physics. Products of k orthonormal basis vectors \mathbf{e}_j can be reduced if two of them are the same, but if they are all distinct, their product is a basis k -vector. In an n -dimensional space, the algebra contains $\binom{n}{k}$ such linearly independent multivectors of *grade* k . The highest-grade element is the volume element, proportional to

$$\mathbf{e}_T \equiv \mathbf{e}_{123\dots n} = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \cdots \mathbf{e}_n .$$

Exercise 26 Find the number of independent elements in the geometric algebra of \mathcal{C}_5 of 5-dimensional space. How is this subdivided into vectors, bivectors, and so on?

Homogeneous Subspaces

A general element of the algebra is a mixture of different grades. We use the notation $\langle x \rangle_k$ to isolate the part of x with grade k . Thus, $\langle x \rangle_0$ is the scalar part of x , $\langle x \rangle_1$ is the vector part, and by $\langle x \rangle_{2,1}$ we mean the sum $\langle x \rangle_2 + \langle x \rangle_1$ of the bivector and vector parts. Evidently

$$x = \langle x \rangle_{0,1,2,\dots,n} = \sum_{k=0}^n \langle x \rangle_k .$$

The elements of each grade k in \mathcal{C}_n form a *homogeneous linear subspace* $\langle \mathcal{C}_n \rangle_k$ of the algebra.

The exterior product of k vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, is the k -grade part of their product:

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_k \equiv \langle \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k \rangle_k .$$

It represents the k -volume contained in the k -dimensional polygon with parallel edges given by the vector factors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, and it vanishes

unless all k vectors are linearly independent. In APS, in addition to scalars, vectors, and bivectors, there are also trivectors, elements of grade 3:

$$\begin{aligned}
\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} &\equiv \langle \mathbf{uvw} \rangle_3 \\
&= \sum_{jkl} u^j v^k w^l \langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 \\
&= \sum_{jkl} \varepsilon_{jkl} u^j v^k w^l \mathbf{e}_{123} \\
&= \mathbf{e}_T \det \begin{pmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{pmatrix}, \tag{2.6}
\end{aligned}$$

where we noted that in 3-dimensional space, the 3-vectors $\langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3$ are all proportional to the volume element $\mathbf{e}_{123} = \mathbf{e}_T$:

$$\langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 = \varepsilon_{jkl} \mathbf{e}_{123},$$

and where ε_{jkl} is the *Levi-Civita symbol*. Note the appearance of the determinant in expression (2.6). It ensures that the wedge product vanishes if the vector factors are linearly dependent.

While the component expressions can be useful for comparing results with other work, the component-free versions $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \equiv \langle \mathbf{uvw} \rangle_3$ are simpler and more efficient to work with. In the trivector $\langle \mathbf{uvw} \rangle_3$, the factor \mathbf{uv} can be split into scalar (grade-0) and bivector (grade-2) parts $\mathbf{uv} = \langle \mathbf{uv} \rangle_0 + \langle \mathbf{uv} \rangle_2$, but $\langle \mathbf{uv} \rangle_0 \mathbf{w}$ is a (grade-1) vector, so that only the bivector piece contributes:

$$\langle \mathbf{uvw} \rangle_3 = \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3.$$

Now split \mathbf{w} into components coplanar with $\langle \mathbf{uv} \rangle_2$ and orthogonal to it:

$$\mathbf{w} = \mathbf{w}^\Delta + \mathbf{w}^\perp,$$

and recall that \mathbf{w}^Δ and \mathbf{w}^\perp anticommute and commute with $\langle \mathbf{uv} \rangle_2$, respectively. The coplanar part \mathbf{w}^Δ is linearly dependent on \mathbf{u} and \mathbf{v} and therefore does not contribute to the trivector. We are left with

$$\begin{aligned}
\langle \mathbf{uvw} \rangle_3 &= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3 = \langle \langle \mathbf{uv} \rangle_2 \mathbf{w}^\perp \rangle_3 \\
&= \frac{1}{2} (\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} + \mathbf{w} \langle \mathbf{uv} \rangle_2) \\
&= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\Im}.
\end{aligned}$$

Similarly, the vector part of the product $\langle \mathbf{uv} \rangle_2 \mathbf{w}$ is

$$\begin{aligned}
\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_1 &= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w}^\Delta \rangle_1 = \frac{1}{2} (\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} - \mathbf{w} \langle \mathbf{uv} \rangle_2) \\
&= \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\Re}.
\end{aligned}$$

Exercise 27 Show that while $\langle \mathbf{uvw} \rangle_3 = \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_3$, the difference $\langle \mathbf{uvw} \rangle_1 - \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_1 = \langle \langle \mathbf{uv} \rangle_0 \mathbf{w} \rangle_1$ does not generally vanish.

A couple of important results follow easily.

Theorem 28 $\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\mathfrak{R}} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) - \mathbf{v}(\mathbf{u} \cdot \mathbf{w})$.

Proof. Expand $\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\mathfrak{R}} = \frac{1}{2}(\langle \mathbf{uv} \rangle_2 \mathbf{w} - \mathbf{w} \langle \mathbf{uv} \rangle_2)$, add and subtract term \mathbf{uwv} and \mathbf{vwu} , and collect:

$$\begin{aligned} \langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\mathfrak{R}} &= \frac{1}{4}[(\mathbf{uv} - \mathbf{vu})\mathbf{w} - \mathbf{w}(\mathbf{uv} - \mathbf{vu})] \\ &= \frac{1}{4}[\mathbf{uvw} + \mathbf{uwv} - \mathbf{vuw} - \mathbf{vwu} - \mathbf{wuv} - \mathbf{uwv} + \mathbf{wvu} + \mathbf{vwu}] \\ &= \frac{1}{2}[\mathbf{u}(\mathbf{v} \cdot \mathbf{w}) - \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u}] \\ &= \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) - \mathbf{v}(\mathbf{u} \cdot \mathbf{w}). \end{aligned}$$

■

Let \mathbf{B} be the bivector $\langle \mathbf{uv} \rangle_2$. If we expand $\mathbf{u} = u^j \mathbf{e}_j$ and $\mathbf{v} = v^k \mathbf{e}_k$, we find

$$\mathbf{B} = u^j v^k \langle \mathbf{e}_j \mathbf{e}_k \rangle_2 = \frac{1}{2} B^{jk} \langle \mathbf{e}_j \mathbf{e}_k \rangle_2,$$

where $B^{jk} = u^j v^k - u^k v^j$ and we noted the antisymmetry of $\langle \mathbf{e}_j \mathbf{e}_k \rangle_2 = -\langle \mathbf{e}_k \mathbf{e}_j \rangle_2$. From the last theorem, the vector $\langle \mathbf{Bw} \rangle_1$ lies in the plane of \mathbf{B} and is orthogonal to \mathbf{w} . In terms of components,

$$\begin{aligned} \langle \mathbf{Bw} \rangle_1 &= \frac{1}{2} B^{jk} w^l \langle \langle \mathbf{e}_j \mathbf{e}_k \rangle_2 \mathbf{e}_l \rangle_1 \\ &= \frac{1}{2} B^{jk} w^l (\mathbf{e}_j \delta_{kl} - \mathbf{e}_k \delta_{jl}) \\ &= B^{jl} w_l \end{aligned}$$

is a *contraction* of the bivector \mathbf{B} with the vector \mathbf{w} and is sometimes written as the dot product $\langle \mathbf{Bw} \rangle_1 = \mathbf{B} \cdot \mathbf{w}$. It lies in the *intersection* of the plane of \mathbf{B} with the hypersurface orthogonal to \mathbf{w} .

Since a vector \mathbf{v} orthogonal to the space spanned by the vectors comprising a k -vector \mathbf{K} commutes with \mathbf{K} if k is even and anticommutes with it if k is odd, we can generalize the result for the trivector to

Theorem 29 The exterior (wedge) product $\langle \mathbf{Kv} \rangle_{k+1}$ of a k -vector \mathbf{K} with a vector \mathbf{v} is given by

$$\langle \mathbf{Kv} \rangle_{k+1} \equiv \mathbf{K} \wedge \mathbf{v} = \frac{1}{2} \left(\mathbf{Kv} + (-)^k \mathbf{vK} \right).$$

Corollary 30 The contraction is $\langle \mathbf{Kv} \rangle_{k-1} \equiv \mathbf{K} \cdot \mathbf{v} = \frac{1}{2} \left(\mathbf{Kv} - (-)^k \mathbf{vK} \right)$.

Duals

Note that in the algebra of an n -dimensional space, the number of independent k -grade multivectors is the same as the number of $(n - k)$ -grade elements. Thus, both the vectors (grade 1 elements) and the pseudovectors (grade $n - 1$ elements) occupy linear spaces of n dimensions. We can therefore establish a one-to-one mapping between such elements. We define the Clifford-Hodge dual $*x$ of an element x by

$$*x \equiv x\mathbf{e}_T^{-1}.$$

The dual of a dual is $\mathbf{e}_T^{-2} = \pm 1$ times the original element. If x is a k -vector, each term in a k -vector basis expansion of x will cancel k of the basis vector factors in \mathbf{e}_T , leaving $*x = x\mathbf{e}_T^{-1}$ as an $(n - k)$ -vector.

A *simple* k -vector is a single product of k independent vectors that span a k -dimensional subspace. Every vector in that subspace is orthogonal to vectors whose products comprise the $(n - k)$ -vector $*x$. In this sense, any simple element and its dual are fully orthogonal. The dual of a scalar is a volume element, known as a *pseudoscalar*; the dual of a vector is the hypersurface orthogonal to that vector, known as a *pseudovector*; and so on.⁴

In physical space ($n = 3$), the dual to a bivector is the vector normal to the plane of the bivector. Thus, $\mathbf{e}_T = \mathbf{e}_{123}$ and $\mathbf{e}_T^{-1} = -\mathbf{e}_{123} = -\mathbf{e}_T$, and for example

$$*(\mathbf{e}_{12}) = \mathbf{e}_{12}(-\mathbf{e}_{123}) = \mathbf{e}_3.$$

We recognize this as the *cross product*:

$$*(\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} \times \mathbf{v},$$

when it is taken between polar vectors, and we can understand its relation to the plane of \mathbf{u} and \mathbf{v} . The volume element in physical space squares to -1 and commutes with all vectors and hence all elements. It can therefore be associated with the *unit imaginary*:

$$\mathbf{e}_T = \mathbf{e}_{123} = i$$

and thus $*(\mathbf{u} \wedge \mathbf{v}) = (\mathbf{u} \wedge \mathbf{v})/i$ so that

$$\mathbf{u} \wedge \mathbf{v} = i\mathbf{u} \times \mathbf{v}.$$

However, whereas the cross product is defined only in three dimensions and is nonassociative as well as noncommutative, the exterior wedge product

⁴These names are most suitable in APS where the pseudoscalar belongs to the center of the algebra, but they are also applied in cases with an even number n of dimensions, where \mathbf{e}_T anticommutes with the vectors.

is defined in spaces of any dimension and is associative. It also emphasizes the essential properties of the plane and is an operator on vectors that generates rotations.⁵

Exercise 31 Verify by calculation of some explicit values that the Levi-Civita symbol is the dual to the volume element $\langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3$ in APS:

$$\varepsilon_{jkl} = * \langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 = \langle \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l \rangle_3 \mathbf{e}_T^{-1}.$$

This definition is easily extended to higher dimensions.

We can use duals to express a rotor in physical space in terms of the *axis of rotation*, which is dual to the rotation plane. For example

$$R = \exp(\mathbf{e}_2 \mathbf{e}_1 \theta / 2) = \exp(-i \mathbf{e}_3 \theta / 2)$$

is the rotor for a rotation $\mathbf{v} \rightarrow R \mathbf{v} R^{-1}$ by θ about the \mathbf{e}_3 axis in physical space. Furthermore, in APS the volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$, is dual to a pseudoscalar, namely the trivector

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} &= \langle \mathbf{abc} \rangle_3 = \langle \mathbf{abc} \rangle_{\mathfrak{S}} \\ &= \langle \langle \mathbf{ab} \rangle_2 \mathbf{c} \rangle_{\mathfrak{S}} = \langle i * \langle \mathbf{ab} \rangle_2 \mathbf{c} \rangle_{\mathfrak{S}} = i \langle (\mathbf{a} \times \mathbf{b}) \mathbf{c} \rangle_{\mathfrak{R}} \\ &= i (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \end{aligned}$$

Exercise 32 The bivector rotation rate $\mathbf{\Omega} = -i \boldsymbol{\omega}$ can be expressed as the dual of a vector $\boldsymbol{\omega}$ in physical space. Show that $\langle \mathbf{\Omega} \mathbf{r} \rangle_{\mathfrak{R}} = \boldsymbol{\omega} \times \mathbf{r}$.

Exercise 33 Show that in physical space the theorem $\langle \langle \mathbf{uv} \rangle_2 \mathbf{w} \rangle_{\mathfrak{R}} = \mathbf{u} (\mathbf{v} \cdot \mathbf{w}) - \mathbf{v} (\mathbf{u} \cdot \mathbf{w})$ is equivalent to $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \cdot \mathbf{w} \mathbf{v} - \mathbf{v} \cdot \mathbf{w} \mathbf{u}$.

Exercise 34 Rewrite the transformation (2.5) from the rotating frame to the inertial lab frame in terms of $\boldsymbol{\omega} = i \mathbf{\Omega}$.

Reciprocal Basis

Except for their normalization, reciprocal basis vectors are duals to hyperplanes formed by wedging all but one of the basis vectors. The reciprocal

⁵The algebra of physical space (APS) thus automatically incorporates complex numbers as its *center* (commuting part). The unit imaginary has geometric meaning in the algebra: it is the unit volume element, which defines the dual relationship. This helps make sense of some of the many complex numbers that appear in *real* physics, and the dual relationship helps avoid confusion associated with pseudoscalars, pseudovectors, and their behavior under inversion. The bivector in APS, for example, is a pseudovector, the dual to the vector normal to the plane of the bivector:

$$\mathbf{e}_{12} = \mathbf{e}_{123} \mathbf{e}_3 = i \mathbf{e}_3.$$

basis is important when the basis is not orthogonal and not necessarily normalized, as in the study of crystalline solids. Thus, if we form a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ from three non-coplanar vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, in APS, the reciprocal vector to \mathbf{a}_1 is

$$\mathbf{a}^1 \equiv \frac{*(\mathbf{a}_2 \wedge \mathbf{a}_3)}{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{\mathbf{a}_2 \wedge \mathbf{a}_3}{\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3} = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)},$$

where we noted that $*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)$ is a real scalar, so that

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{a}^1 &= \frac{\mathbf{a}_1 \cdot *(\mathbf{a}_2 \wedge \mathbf{a}_3)}{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)}{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = 1 \\ \mathbf{a}_2 \cdot \mathbf{a}^1 &= \frac{\mathbf{a}_2 \cdot *(\mathbf{a}_2 \wedge \mathbf{a}_3)}{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = \frac{*(\mathbf{a}_2 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)}{*(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3)} = 0 = \mathbf{a}_3 \cdot \mathbf{a}^1 . \end{aligned}$$

We can think of the reciprocal vectors as basis 1-forms, that is linear operators \mathbf{a}^k on vectors \mathbf{a}_j whose operation is defined by

$$\mathbf{a}^k(\mathbf{a}_j) = \mathbf{a}_j \cdot \mathbf{a}^k = \delta_j^k .$$

3 Paravectors and Relativity

The space of scalars, the space of vectors, and the space of bivectors, are all linear subspaces of the full 2^n -dimensional space of the algebra \mathcal{C}_n . Direct sums of the subspaces are also linear subspaces of the algebra. The most important is the direct sum of the scalar and the vector subspaces. It is an $(n+1)$ -dimensional linear space known as *paravector space*. In APS ($n=3$), every element reduces to a complex paravector.

Elements of real paravector space have the form $p = p^0 + \mathbf{p} = \langle p \rangle_0 + \langle p \rangle_1$, and the algebra \mathcal{C}_n also includes their exterior products:

$$\begin{aligned} \text{paravector space} &= \langle \mathcal{C}_n \rangle_{1,0} , (n+1)\text{-dim} \\ \text{biparavector space} &= \langle \mathcal{C}_n \rangle_{2,1} , \binom{n+1}{2}\text{-dim} \\ \text{k-paravector space} &= \langle \mathcal{C}_n \rangle_{k,k-1} , \binom{n+1}{k}\text{-dim} . \end{aligned}$$

In general, grade-0 paravectors are scalars in $\langle \mathcal{C}_n \rangle_0$, $(n+1)$ -grade paravectors are volume elements (pseudoscalars) in $\langle \mathcal{C}_n \rangle_n$, and the linear space of k -grade multiparavectors is $\langle \mathcal{C}_n \rangle_{k,k-1} \equiv \langle \mathcal{C}_n \rangle_k \oplus \langle \mathcal{C}_n \rangle_{k-1}$, $k = 1, 2, \dots, n$.

3.1 Clifford Conjugation

In addition to reversal (dagger-conjugation), introduced above, we need Clifford conjugation, the anti-automorphism that changes the sign of all

vector factors as well as reversing their order and their combination. For any paravector p , its Clifford conjugate is

$$\bar{p} = p^0 - \mathbf{p}, \quad \overline{pq} = \bar{q}\bar{p}$$

Clifford conjugation can be used to split elements into *scalarlike* and *vectorlike* parts:

$$p = \frac{p + \bar{p}}{2} + \frac{p - \bar{p}}{2} = \langle p \rangle_S + \langle p \rangle_V = \text{scalarlike} + \text{vectorlike}.$$

Clifford conjugation is combined with reversal to give the *grade automorphism*

$$\bar{p}^\dagger = \bar{p}, \quad (\overline{pq})^\dagger = \overline{(pq)^\dagger} = \bar{p}^\dagger \bar{q}^\dagger,$$

with which elements can be split into *even* and *odd* parts:

$$p = \frac{p + \bar{p}^\dagger}{2} + \frac{p - \bar{p}^\dagger}{2} = \langle p \rangle_+ + \langle p \rangle_- = \text{even} + \text{odd}.$$

These relations offer simple ways to isolate different vector and paravector grades. In particular, for $n = 3$, (here \dots stands for any expression)

$$\begin{aligned} \langle \dots \rangle_S &= \langle \dots \rangle_{0,3} & \langle \dots \rangle_V &= \langle \dots \rangle_{1,2} \\ \langle \dots \rangle_{\mathfrak{R}} &= \langle \dots \rangle_{0,1} & \langle \dots \rangle_{\mathfrak{S}} &= \langle \dots \rangle_{2,3} \\ \langle \dots \rangle_+ &= \langle \dots \rangle_{0,2} & \langle \dots \rangle_- &= \langle \dots \rangle_{1,3} . \end{aligned}$$

Exercise 35 Verify that the splits can be combined to extract individual vector grades as follows:

$$\begin{aligned} \langle \dots \rangle_0 &= \langle \dots \rangle_{\mathfrak{RS}} = \langle \dots \rangle_{\mathfrak{R}+} = \langle \dots \rangle_{S+} \\ \langle \dots \rangle_1 &= \langle \dots \rangle_{\mathfrak{RV}} \\ \langle \dots \rangle_2 &= \langle \dots \rangle_{\mathfrak{SV}} \\ \langle \dots \rangle_3 &= \langle \dots \rangle_{\mathfrak{SS}} . \end{aligned}$$

Example 36 Let \mathbf{B} be any bivector. Then \mathbf{B} is even, imaginary, and vectorlike, whereas \mathbf{B}^2 is even, real, and scalarlike. Any analytic function $f(\mathbf{B})$ is even and $\overline{f(\mathbf{B})} = f(-\mathbf{B}) = f(\bar{\mathbf{B}}) = f(\mathbf{B}^\dagger)$. Spatial rotors $R(\mathbf{B}) = \exp(\mathbf{B}/2)$ are even and unitary: $R^\dagger(\mathbf{B}) = R^{-1}(\mathbf{B}) = R(-\mathbf{B})$.⁶

Exercise 37 Show that the wedge (exterior) and dot (contraction) products of an arbitrary element x with a vector \mathbf{v} in $\mathcal{C}\ell_n$ is given by

$$\begin{aligned} x \wedge \mathbf{v} &= \frac{1}{2} (x\mathbf{v} + \mathbf{v}\bar{x}^\dagger) \\ x \cdot \mathbf{v} &= \frac{1}{2} (x\mathbf{v} - \mathbf{v}\bar{x}^\dagger) . \end{aligned}$$

⁶We can use either grade numbers or conjugation symmetries to split an element into parts. The grade numbers emphasize the algebraic structure whereas the conjugation symmetries indicate an operational procedure to compute the part.

3.2 Paravector Metric

If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal basis of the original Euclidean space, so that

$$\langle \mathbf{e}_j \mathbf{e}_k \rangle_0 = \frac{1}{2} (\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j) = \delta_{jk} ,$$

the proper basis of paravector space is $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where we identify $\mathbf{e}_0 \equiv 1$ for convenience in expanding paravectors $p = p^\mu \mathbf{e}_\mu$, $\mu = 0, 1, \dots, n$ in the basis. The metric of paravector space is determined by a quadratic form. We need a product of a paravector p with itself or a conjugate that is scalar valued. It is easy to see that p^2 generally has vector parts, but $p\bar{p} = \langle p\bar{p} \rangle_0 = \bar{p}p$ is a scalar. Therefore it is adopted as the *quadratic form* (“square length”):

$$Q(p) = p\bar{p}.$$

By “polarization” $p \rightarrow p + q$ we find the inner product

$$\begin{aligned} \langle p, q \rangle &= \langle p\bar{q} \rangle_S = \frac{1}{2} (p\bar{q} + q\bar{p}) \\ &= p^\mu q^\nu \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_S \equiv p^\mu q^\nu \eta_{\mu\nu} . \end{aligned}$$

Exercise 38 Show that the inner product $\langle p\bar{q} \rangle_S = \frac{1}{2} [Q(p+q) - Q(p) - Q(q)]$.

Exercise 39 Find the values of $\eta_{\mu\nu} = \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_S$.

We recognize the matrix

$$(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, \dots, -1)$$

as the natural metric of the paravector space. It has the form of the *Minkowski metric*. If $\langle p\bar{q} \rangle_S = 0$, then the paravectors p and q are *orthogonal*. For any element x , $x\bar{x} = \overline{x\bar{x}} = \langle x\bar{x} \rangle_S$. In the standard matrix representation of \mathcal{Cl}_3 ,

$$x\bar{x} \simeq \det \underline{x} .$$

If $x\bar{x} = 1$, x is *unimodular*.

The inverse of an element x can be written

$$x^{-1} = \frac{\bar{x}}{x\bar{x}},$$

but this does not exist if $x\bar{x} = 0$. The existence of nonzero elements of zero length means that Clifford’s geometric algebra, unlike the algebras of reals, complexes, and quaternions, is generally *not* a division algebra. This may seem an annoyance at first, but it is the basis for powerful projector techniques, as we demonstrate below.

Exercise 40 Show that the paravector $1 + \mathbf{e}_1$ has no inverse and is *orthogonal to itself*.

Spacetime as Paravector Space

The paravectors of physical space provide a covariant model of spacetime. (Extensions to curved spacetimes are possible, but for simplicity we restrict ourselves here to flat spacetimes.) We use SI units with $c = 1$ and, unless specified otherwise, take $n = 3$. Spacetime vectors are represented by paravectors whose frame-dependent split into scalar and vector parts reflects the observer's ability to distinguish time and space components. In particular, any timelike spacetime displacement $dx = dt + d\mathbf{x}$ has a Lorentz-invariant length defined as the proper time interval $d\tau$:

$$d\tau^2 = dx d\bar{x} = \eta_{\mu\nu} dx^\mu dx^\nu.$$

The proper velocity is

$$u = \frac{dx}{d\tau} = \gamma(1 + \mathbf{v}) = u^\mu \mathbf{e}_\mu$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the usual coordinate velocity and we use the summation convention for repeated indices. Other spacetime vectors are similarly represented, for example,

$$\begin{aligned} p &= mu = E + \mathbf{p} : \text{paramomentum} \\ j &= j^\mu \mathbf{e}_\mu = \rho + \mathbf{j} : \text{current density} \\ A &= A^\mu \mathbf{e}_\mu = \phi + \mathbf{A} : \text{paravector potential} \\ \partial &= \partial_\mu \mathbf{e}_\nu \eta^{\mu\nu} = \partial_t - \nabla : \text{gradient operator.} \end{aligned}$$

Exercise 41 Consider a function $f(s)$, where s is the scalar $s = \langle k\bar{x} \rangle_S = k^\mu x_\mu = k_\mu x^\mu$ and k is a constant paravector. Use the chain rule for differentiation to prove that $\partial f(s) = k f'(s)$, where $f'(s) = df/ds$. Note that $\partial_\mu \equiv \partial/\partial x^\mu$.

Biparavectors represent *oriented planes* in *spacetime*, for example the electromagnetic field

$$\mathbf{F} = \langle \partial \bar{A} \rangle_V = \frac{1}{2} F^{\mu\nu} \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_V = \mathbf{E} + i\mathbf{B}.$$

The basis biparavectors $\langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_V = -\langle \mathbf{e}_\nu \bar{\mathbf{e}}_\mu \rangle_V$ are also the generators of Lorentz rotations, and the expansion of \mathbf{F} in this basis gives directly the usual tensor components $F^{\mu\nu}$. However, no tensor elements are needed in APS, and the algebra offers several ways to interpret \mathbf{F} geometrically. The field at any point is a covariant plane in spacetime, and for any observer it splits naturally into frame-dependent parts: a timelike plane $\mathbf{E} = \mathbf{E}\mathbf{e}_0 = \langle \mathbf{F} \rangle_1$ and a spatial plane $i\mathbf{B} = \langle \mathbf{F} \rangle_2$. Since $\mathbf{E} = \langle \mathbf{F}\mathbf{e}_0 \rangle_{\mathfrak{R}}$, the electric field \mathbf{E} lies in the intersection of \mathbf{F} with the spatial hyperplane dual (and thus orthogonal) to \mathbf{e}_0 . From $i\mathbf{B} = \langle \mathbf{F}\mathbf{e}_0 \rangle_{\mathfrak{S}}$, we see that the usual magnetic field \mathbf{B} is the vector dual to the spacetime hypersurface $\langle \mathbf{F}\mathbf{e}_0 \rangle_{\mathfrak{S}}$.

We need to be careful about stating that \mathbf{F} is a plane in spacetime. The sum of electromagnetic fields is another electromagnetic field, but the sum of planes in four dimensions is not necessarily a single plane. We may get two orthogonal planes. (Of course this cannot happen in three dimensions, where the sum of any two planes is also a plane, but spacetime has four dimensions.) Thus, we distinguish simple fields, which are single spacetime planes from compound ones, which occupy two orthogonal (and hence commuting) planes. How do we distinguish a simple field from a compound one? All we need to do is square it. The square of any simple field is a real scalar, whereas the square of a compound field contains a spacetime volume element, that is a pseudoscalar, given in APS as an imaginary scalar.

Example 42 *The bivector $\langle p\bar{q} \rangle_V = \frac{1}{2}(p\bar{q} - q\bar{p})$ is simple and represents a spacetime plane containing paravectors p and q . Its square*

$$\begin{aligned} \langle p\bar{q} \rangle_V^2 &= \frac{1}{4}(p\bar{q} - q\bar{p})^2 = \frac{1}{4}(p\bar{q} + q\bar{p})^2 - \frac{1}{2}(p\bar{q}q\bar{p} + q\bar{p}p\bar{q}) \\ &= \langle p\bar{q} \rangle_0^2 - p\bar{p}q\bar{q} \end{aligned}$$

is seen to be a real scalar.

If paravectors p and q are orthogonal to both r and s , then $\langle p\bar{q} + r\bar{s} \rangle_V$ is a compound bivector with orthogonal spacetime planes $\langle p\bar{q} \rangle_V$ and $\langle r\bar{s} \rangle_V$. However, the planes in a compound bivector may not be unique: if p, q, r, s are mutually orthogonal paravectors and if $p\bar{p} = r\bar{r}$ and $q\bar{q} = s\bar{s}$, then

$$\langle p\bar{q} + r\bar{s} \rangle_V = \frac{1}{2} \left\langle (p+r)\overline{(q+s)} + (p-r)\overline{(q-s)} \right\rangle_V$$

expresses the compound bivector as two sets of orthogonal planes. Orthogonal planes in spacetime are proportional to each other's dual. As a result, any compound bivector can be expressed as a simple bivector times a complex number.

Exercise 43 *Show that the compound bivector $\mathbf{e}_1\bar{\mathbf{e}}_0 + \mathbf{e}_3\bar{\mathbf{e}}_2 = \mathbf{e}_1\bar{\mathbf{e}}_0(1+i)$.*

The square of $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ is

$$\mathbf{F}^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E} \cdot \mathbf{B}$$

and \mathbf{F} is evidently simple if and only if $\mathbf{E} \cdot \mathbf{B} = 0$. A null field, with $\mathbf{F}^2 = 0$, is thus simple. It can be written $\mathbf{F} = (1 + \hat{\mathbf{k}})\mathbf{E}$, where $\hat{\mathbf{k}}\mathbf{E} = i\mathbf{B}$.

Exercise 44 *Show that any null field $\mathbf{F} = (1 + \hat{\mathbf{k}})\mathbf{E}$ obeys $\hat{\mathbf{k}}\mathbf{F} = \mathbf{F} = -\mathbf{F}\hat{\mathbf{k}}$.*

Lorentz Transformations

Much of the power of Clifford's geometric algebra in relativistic applications arises from the form of Lorentz transformations. In APS, we identify paravector space with spacetime, and physical (restricted) Lorentz transformations are rotations in paravector space. They take the form of spin transformations

$$\begin{aligned} p &\rightarrow LpL^\dagger, \text{ odd multiparavector grade} \\ \mathbf{F} &\rightarrow L\mathbf{F}\bar{L}, \text{ even multiparavector grade,} \end{aligned}$$

where the Lorentz rotors L are unimodular ($L\bar{L} = 1$) and have the form

$$\begin{aligned} L &= \pm \exp(\mathbf{W}/2) \in SL(2, \mathbb{C}) \\ \mathbf{W} &= \frac{1}{2}W^{\mu\nu} \langle \mathbf{e}_\mu \bar{\mathbf{e}}_\nu \rangle_{1,2}. \end{aligned}$$

Every L can be factored into a boost $B = B^\dagger$ (a real factor) and a spatial rotation $R = \bar{R}^\dagger$ (a unitary factor): $L = BR$.

For any paravectors p, q , the scalar product $\langle p\bar{q} \rangle_S \rightarrow \langle LpL^\dagger \bar{L}^\dagger \bar{q}\bar{L} \rangle_S = \langle p\bar{q} \rangle_S$ is Lorentz invariant. In particular, the square length $p\bar{p} = (p^0)^2 - \mathbf{p}^2$ is invariant and can be timelike (> 0), spacelike (< 0), or lightlike (null, $= 0$). Null paravectors are orthogonal to themselves. Similarly \mathbf{F}^2 is Lorentz invariant, and simple fields can be classified as predominantly electric ($\mathbf{F}^2 > 0$), predominantly magnetic ($\mathbf{F}^2 < 0$), or null ($\mathbf{F}^2 = 0$).

With Lorentz transformations, we can easily transform properties between inertial frames. The *position coordinate* x of a particle instantaneously *at rest* changes only by its time, the proper time τ : $dx_{\text{rest}} = d\tau$. We transform this to the lab, in which the particle moves with proper velocity $u = dx/d\tau$, by

$$\begin{aligned} dx &= Ldx_{\text{rest}}L^\dagger = LL^\dagger d\tau = u d\tau \\ &= dt + d\mathbf{x} = dt(1 + \mathbf{v}). \end{aligned}$$

With $L = BR$ it follows that

$$LL^\dagger = B^2 = u = \frac{dt}{d\tau}(1 + \mathbf{v}).$$

Now $LL^\dagger = L\mathbf{e}_0L^\dagger = u$ is just the Lorentz rotation of the unit basis paravector \mathbf{e}_0 , and its square length is invariant:

$$u\bar{u} = 1 = \gamma^2(1 - \mathbf{v}^2),$$

where $\gamma = dt/d\tau$ is the *time-dilation factor*.

Example 45 Consider the transformation of a paravector $p = p^\mu \mathbf{e}_\mu$ in a system that is boosted from rest to a velocity $\mathbf{v} = v\mathbf{e}_3$:

$$p \rightarrow LpL^\dagger = BpB = p^\mu \mathbf{u}_\mu$$

where $B = \exp(w\mathbf{e}_3\bar{\mathbf{e}}_0/2) = u^{1/2}$ represents a rotation in the $\mathbf{e}_3\bar{\mathbf{e}}_0$ paravector plane and $\mathbf{u}_\mu = B\mathbf{e}_\mu B$ is the boosted proper basis paravector. Evidently

$$\begin{aligned}\mathbf{u}_0 &= B\mathbf{e}_0B = B^2\mathbf{e}_0 = u\mathbf{e}_0 = \gamma(\mathbf{e}_0 + v\mathbf{e}_3) \\ \mathbf{u}_1 &= \mathbf{e}_1 \\ \mathbf{u}_2 &= \mathbf{e}_2 \\ \mathbf{u}_3 &= B\mathbf{e}_3B = u\mathbf{e}_3 = \gamma(\mathbf{e}_3 + v\mathbf{e}_0)\end{aligned}$$

with $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$.

As in the case of spatial rotations, if we put $LpL^\dagger = p' = p'^\nu\mathbf{e}_\nu$, we can

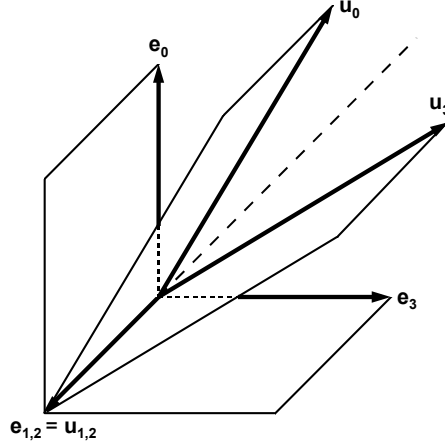


FIGURE 6. Spacetime diagram showing the boost to $\mathbf{v} = 0.6\mathbf{e}_3$.

easily find

$$p'^\nu = p^\mu \langle \mathbf{u}_\mu \bar{\mathbf{e}}^\nu \rangle_S$$

and hence the usual 4×4 matrix relating the components of p before and after the boost, but we have no need of it. The relations for \mathbf{u}_μ are useful for drawing spacetime diagrams. Thus, if $v = 0.6$, then $\gamma = 1.25$ and

$$\begin{aligned}\mathbf{u}_0 &= (5\mathbf{e}_0 + 3\mathbf{e}_3)/4 \\ \mathbf{u}_3 &= (5\mathbf{e}_3 + 3\mathbf{e}_0)/4.\end{aligned}$$

We can take this further and look at planes in spacetime, as shown in Fig. 6.

Exercise 46 Show that the bivectors $\mathbf{e}_3\bar{\mathbf{e}}_0$ and $\mathbf{e}_1\bar{\mathbf{e}}_2$ are invariant under any boost B along \mathbf{e}_3 .

Exercise 47 Let system B have proper velocity u_{AB} with respect to A , and let system C have proper velocity u_{BC} as seen by an observer in B . Show that the proper velocity of C as viewed by A is

$$u_{AC} = u_{AB}^{1/2} u_{BC}^{1/2} u_{AB}$$

and that this reduces to the product $u_{AC} = u_{AB} u_{BC}$ when the spatial velocities are collinear. Writing each proper velocity in the form $u = \gamma(1 + \mathbf{v})$, show that in the collinear case

$$\mathbf{v}_{AC} = \frac{\langle u_{AC} \rangle_V}{\langle u_{AC} \rangle_S} = \frac{\mathbf{v}_{AB} + \mathbf{v}_{BC}}{1 + \mathbf{v}_{AB} \cdot \mathbf{v}_{BC}}.$$

Example 48 Consider a boost of the photon wave paravector

$$k = \omega(1 + \hat{\mathbf{k}}) \rightarrow k' = BkB = u(\omega + \mathbf{k}^{\parallel}) + \mathbf{k}^{\perp}$$

with $\mathbf{k}^{\parallel} = \mathbf{k} \cdot \hat{\mathbf{v}} \hat{\mathbf{v}} = \mathbf{k} - \mathbf{k}^{\perp}$ and $u = \gamma(1 + \mathbf{v})$. This describes what happens to the photon momentum when the light source is boosted. Evidently \mathbf{k}^{\perp} is unchanged, but there is a Doppler shift and a change in \mathbf{k}^{\parallel} :

$$\begin{aligned} \omega' &= \langle u(\omega + \mathbf{k}^{\parallel}) \rangle_S = \gamma\omega(1 + \hat{\mathbf{k}} \cdot \mathbf{v}) \\ \mathbf{k}' \cdot \hat{\mathbf{v}} &= \langle u\hat{\mathbf{v}}(\omega + \mathbf{k}^{\parallel}) \rangle_S = \gamma\omega(v + \hat{\mathbf{k}} \cdot \hat{\mathbf{v}}) = \omega' \cos \theta'. \end{aligned}$$

Thus the photons are thrown forward

$$\cos \theta' = \frac{v + \cos \theta}{1 + v \cos \theta}. \quad (3.1)$$

in what is called the “headlight” effect (see Fig. 7).

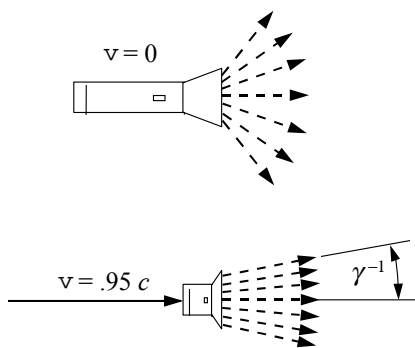


FIGURE 7. Headlight effect in boosted light source.

Exercise 49 Solve Eq. (3.1) for $\cos\theta$ and show that the result is the same as in Eq. (3.1) except that v is replaced by $-v$ and θ and θ' are interchanged.

Exercise 50 Show that at high velocities, the radiation from the boosted source is concentrated in the cone of angle γ^{-1} about the forward direction.

Simple rotors of Lorentz transformations can be expressed as a product of paravectors in the spacetime plane of rotation. Consider a biparavector $\langle p\bar{q} \rangle_V$ and a paravector r with a coplanar component r^Δ and an orthogonal component r^\perp . The coplanar component is a linear combination of p and q : $r^\Delta = \alpha p + \beta q$, where α and β are scalars. Now the product of $\langle p\bar{q} \rangle_V$ with p satisfies

$$\begin{aligned} \langle p\bar{q} \rangle_V p &= \frac{1}{2} (p\bar{q}p - q\bar{p}p) = p \langle \bar{q}p \rangle_V \\ &= p \langle p\bar{q} \rangle_V^\dagger . \end{aligned}$$

Similarly with q and thus with any linear combination of p and q . It follows that if L is a simple Lorentz rotor in the plane $\langle p\bar{q} \rangle_V$, that the coplanar component obeys

$$Lr^\Delta = r^\Delta L^\dagger .$$

On the other hand r^\perp is orthogonal to the plane and thus coplanar with its dual: $\langle p\bar{r}^\perp \rangle_S = 0 = \langle q\bar{r}^\perp \rangle_S$, so that

$$\langle p\bar{q} \rangle_V r^\perp = \frac{1}{2} (p\bar{q}r^\perp - q\bar{p}r^\perp) = \frac{1}{2} r^\perp (\bar{p}q - \bar{q}p) = -r^\perp \langle p\bar{q} \rangle_V^\dagger$$

and consequently for any Lorentz rotor L that rotates in the plane of $\langle p\bar{q} \rangle_V$,

$$\bar{L}r^\perp = r^\perp L^\dagger .$$

The Lorentz transformation of r thus gives

$$LrL^\dagger = r^\perp + L^2 r^\Delta .$$

In spacetime, it is possible for a null vector to be both coplanar and orthogonal to a null flag. An example of a null flag is $\mathbf{F} = (1 + \mathbf{e}_3) \mathbf{e}_1 = (1 + \mathbf{e}_3) \mathbf{e}_3 \mathbf{e}_1 = i(1 + \mathbf{e}_3) \mathbf{e}_2$. The dual flag ${}^*\mathbf{F} = -i\mathbf{F} = (1 + \mathbf{e}_3) \mathbf{e}_2$ is the rotation of \mathbf{F} about \mathbf{e}_3 by $\pi/2$. A Lorentz transformation generated by \mathbf{F} leaves the flagpole $1 + \mathbf{e}_3$ invariant, since $\mathbf{F}(1 + \mathbf{e}_3) = 0 = (1 + \mathbf{e}_3) \mathbf{F}^\dagger$.

Suppose that r lies in the plane of rotation of L and that

$$s = LrL^\dagger = L^2 r .$$

Then, as long as $r\bar{r} \neq 0$,

$$L = (sr^{-1})^{1/2} = \frac{(s+r)r^{-1}}{\sqrt{2\langle sr^{-1}+1 \rangle_S}} = \frac{sr^{-1}+1}{\langle 2(sr^{-1}+1) \rangle_S^{1/2}} .$$

Exercise 51 Verify that $L\bar{L} = 1$ and

$$L^2 r = \frac{(sr^{-1} + 1)(s + r)}{\langle 2(sr^{-1} + 1) \rangle_S} = \frac{2 + sr^{-1} + rs^{-1}}{2 + sr^{-1} + \bar{r}^{-1}\bar{s}} s = s .$$

[Hint: note that $s\bar{s} = r\bar{r} \neq 0$ so that $\bar{r}^{-1}\bar{s} = r\bar{s}/(r\bar{r}) = r\bar{s}/(s\bar{s}) = rs^{-1}$.]

If we rotate a null paravector $k = \omega(1 + \hat{\mathbf{k}})$ in a spacetime plane that contains k , then $k \rightarrow k' = LkL^\dagger = L^2k$. In the case of a boost, $L = B = \exp\left(\frac{w}{2}\hat{\mathbf{k}}\right)$, we find

$$k' = e^w k$$

with $e^w = \langle k' \rangle_S / \langle k \rangle_S = \omega' / \omega = \gamma(1 + v) = \sqrt{(1 + v)/(1 - v)}$.

Lorentz rotations in the same or in dual planes commute, but otherwise they generally do not. Furthermore, whereas any product of spatial rotations is another spatial rotation, the product of noncommuting boosts generally does not give a pure boost, but rather the product of a boost and a rotation. Lorentz rotations can also be expressed as the product of spacetime reflections. Up to four reflections may be needed.

3.3 Relation of APS to STA

An alternative to the paravector model of spacetime in APS is the spacetime algebra (STA) introduced by David Hestenes[10]. They are closely related, and it is the purpose of this section to show how.

STA is the geometric algebra $\mathcal{C}\ell_{1,3}$ of Minkowski spacetime. It starts with a 4-dimensional basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \equiv \{\gamma_\mu\}$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$$

in each frame. The chosen frame can be independent of the observer and her frame $\{\hat{\gamma}_\mu\}$. Any spacetime vector $p = p^\mu \hat{\gamma}_\mu$ can be multiplied by $\hat{\gamma}_0$ to give the *spacetime split*

$$p\hat{\gamma}_0 = p \cdot \hat{\gamma}_0 + p \wedge \hat{\gamma}_0 = p^0 + p^k \mathbf{e}_k ,$$

where $\mathbf{e}_0 = 1$ and $\mathbf{e}_k \equiv \hat{\gamma}_k \hat{\gamma}_0$ are the proper basis paravectors of the system in APS. (This association establishes the previously mentioned isomorphism between the even subalgebra of STA and APS.) More general paravector basis elements \mathbf{u}_μ in APS arise when the basis $\{\gamma_\mu\}$ used for the expansion $p = p^\mu \gamma_\mu$ is for a frame in motion with respect to the observer:⁷

$$\mathbf{u}_\mu = \gamma_\mu \hat{\gamma}_0 .$$

⁷A double arrow might be thought more appropriate than an equality here, because \mathbf{u}_μ and $\gamma_\mu, \hat{\gamma}_0$ act in different algebras. However, we are identifying $\mathcal{C}\ell_3$ with the even

In particular, $\mathbf{u}_0 = \gamma_0 \hat{\gamma}_0$ is the proper velocity of the frame $\{\gamma_\mu\}$ with respect to the observer frame $\{\hat{\gamma}_\mu\}$. The basis vectors in APS are relative; they always relate two frames, but those in STA can be considered absolute.

Clifford conjugation in APS corresponds to reversion in STA, indicated by a tilde:

$$\bar{\mathbf{u}}_\mu = (\gamma_\mu \hat{\gamma}_0)^\sim = \hat{\gamma}_0 \gamma_\mu .$$

For example, if the proper velocity of frame $\{\gamma_\mu\}$ with respect to $\hat{\gamma}_0$ is $\mathbf{u}_0 = \gamma_0 \hat{\gamma}_0$, then the proper velocity of frame $\{\hat{\gamma}_\mu\}$ with respect to γ_0 is $\bar{\mathbf{u}}_0 = \hat{\gamma}_0 \gamma_0$. It is not possible to make all of the basis vectors in any STA frame Hermitian, but one usually takes $\hat{\gamma}_0^\dagger = \hat{\gamma}_0$ and $\hat{\gamma}_k^\dagger = -\hat{\gamma}_k$ in the observer's frame $\{\hat{\gamma}_\mu\}$. Hermitian conjugation in STA then combines reversion with the reflection in the observer's time axis $\hat{\gamma}_0$: $\Gamma^\dagger = \hat{\gamma}_0 \tilde{\Gamma} \hat{\gamma}_0$, for example

$$\begin{aligned} \mathbf{u}_\mu^\dagger &= \left[\hat{\gamma}_0 (\gamma_\mu \hat{\gamma}_0)^\sim \hat{\gamma}_0 \right] \\ &= \gamma_\mu \hat{\gamma}_0 = \mathbf{u}_\mu , \end{aligned}$$

which shows that all the paravector basis vectors \mathbf{u}_μ are Hermitian. It is important to note that Hermitian conjugation is frame dependent in STA just as Clifford conjugation of paravectors is in APS.

Example 52 *The Lorentz-invariant scalar part of the paravector product $p\bar{q}$ in APS thus becomes*

$$\begin{aligned} \langle p\bar{q} \rangle_S &= \frac{1}{2} p^\mu q^\nu (\mathbf{e}_\mu \bar{\mathbf{e}}_\nu + \mathbf{e}_\nu \bar{\mathbf{e}}_\mu) \\ &= \frac{1}{2} p^\mu q^\nu (\gamma_\mu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\nu + \gamma_\nu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\mu) \\ &= p^\mu q^\nu \eta_{\mu\nu} . \end{aligned}$$

Biparavector basis elements in APS become basis bivectors in STA:

$$\begin{aligned} \frac{1}{2} (\mathbf{e}_\mu \bar{\mathbf{e}}_\nu - \mathbf{e}_\nu \bar{\mathbf{e}}_\mu) &= \frac{1}{2} (\gamma_\mu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\nu - \gamma_\nu \hat{\gamma}_0 \hat{\gamma}_0 \gamma_\mu) \\ &= \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) . \end{aligned}$$

Lorentz transformations in STA are effected by

$$\gamma_\mu \rightarrow L \gamma_\mu L^\sim ,$$

subalgebra of $\mathcal{C}\ell_{1,3}$, so that the one algebra is embedded in the other. Caution is still needed to avoid statements such as

$$i = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \stackrel{\text{wrong!}}{=} \hat{\gamma}_1 \wedge \hat{\gamma}_0 \wedge \hat{\gamma}_2 \wedge \hat{\gamma}_0 \wedge \hat{\gamma}_3 \wedge \hat{\gamma}_0 = 0 .$$

This is not valid because the wedge products on either side of the third equality refer to different algebras and are not equivalent.

with $LL^\sim = 1$. Every product of basis vectors transforms the same way. An *active transformation* keeps the observer frame fixed and transforms only the system frame:

$$\begin{aligned} \mathbf{e}_\mu = \hat{\gamma}_\mu \hat{\gamma}_0 \rightarrow \mathbf{u}_\mu &= \gamma_\mu \hat{\gamma}_0 = L \hat{\gamma}_\mu L^\sim \hat{\gamma}_0 = L \hat{\gamma}_\mu \hat{\gamma}_0 (\hat{\gamma}_0 L^\sim \hat{\gamma}_0) \\ &= L \mathbf{e}_\mu L^\dagger. \end{aligned}$$

We noted that the $\hat{\gamma}_0$ in the definition of \mathbf{e}_μ is always the observer's time axis. In a *passive transformation*, it is the system frame that stays the same and the observer's frame that changes. Let us suppose that the observer moves from frame $\{\hat{\gamma}_\mu\}$ to frame $\{\hat{\gamma}'_\mu\}$ where $\gamma'_\mu = L \hat{\gamma}'_\mu L^\sim$. Then

$$\mathbf{e}_\mu = \gamma'_\mu \hat{\gamma}_0 \rightarrow \mathbf{u}_\mu = \gamma'_\mu \hat{\gamma}'_0.$$

To re-express the transformed relative coordinates \mathbf{u}_μ in terms of the original \mathbf{e}_μ , we must expand the system frame vectors γ'_μ in terms of the observer's transformed basis vectors. Thus

$$\mathbf{u}_\mu = L \hat{\gamma}'_\mu L^\sim \hat{\gamma}_0 = L \mathbf{e}_\mu L^\dagger.$$

The mathematics is identical to that for the active transformation, but the interpretation is different. Since the transformations can be realized by changing the observer frame and keeping the system frame constant, the physical objects can be taken to be fixed in STA, giving what is referred to as an *invariant* treatment of relativity.

The Lorentz rotation is the same whether we rotate the object forward or the observer backward or some combination. This is trivially seen in APS where only the object frame relative to the observer enters. Furthermore, as seen above, the space/time split of a property in APS is simply a result of expanding into vector grades in the observer's proper basis $\{\mathbf{e}_\mu\}$:

$$\begin{aligned} p &= \langle p \rangle_0 + \langle p \rangle_1 = p^0 + \mathbf{p} \\ \mathbf{F} &= \langle \mathbf{F} \rangle_1 + \langle \mathbf{F} \rangle_2 = \mathbf{E} + i\mathbf{B}. \end{aligned}$$

To get the split for a different observer, you can expand p in his paravector basis $\{\mathbf{u}_\mu\}$ and \mathbf{F} in his biparavector basis $\{\langle \mathbf{u}_\mu \bar{\mathbf{u}}_\nu \rangle_{1,2}\}$, where $u = \mathbf{u}_0$ is his proper velocity relative to the original observer. Then with the transformation $\mathbf{u}_\mu = L \mathbf{e}_\mu L^\dagger$ you re-express the result in her proper basis before splitting vector grades. The physical fields, momenta, etc. are transformed and are not invariant in APS, but *covariant*, that is the form of the equations remains the same but not the vectors and multivectors themselves.⁸

⁸You can have *absolute frames* in APS, if you want them for use in passive transformations, by introducing an *absolute observer*.

4 Eigenspinors

A Lorentz rotor of particular interest is the *eigenspinor* Λ that relates the particle reference frame to the observer. It transforms distinctly from paravectors and their products:

$$\Lambda \rightarrow L\Lambda$$

and is a generally reducible element of the spinor carrier space of Lorentz rotations $\in SL(2, \mathbb{C})$. This property makes Λ a *spinor*. The *eigen* part refers to its association with the particle. Indeed any property of the particle in the reference frame is easily transformed by Λ to the lab (= observer's frame). For example, the proper velocity of a massive particle can be taken to be $u = 1$ in the reference frame. In the lab it is then

$$u = \Lambda \mathbf{e}_0 \Lambda^\dagger = \Lambda \Lambda^\dagger,$$

which is seen to be the timelike basis vector of a frame moving with proper velocity u (with respect to the observer).

If we write $\Lambda = BR$, then u is independent of R . Traditional particle dynamics gives only u and by integration the world line. The eigenspinor gives more, namely the orientation and the full moving frame $\{\mathbf{u}_\mu = \Lambda \mathbf{e}_\mu \Lambda^\dagger\}$. While \mathbf{u}_0 is the proper velocity, \mathbf{u}_3 is essentially the Pauli-Lubański spin paravector.[11]

4.1 Time Evolution

The eigenspinor Λ changes in time, with $\Lambda(\tau)$ giving the Lorentz rotation at time τ . For boosts (rotations in timelike planes) this means that Λ relates the observer frame to the commoving inertial frame of the object at τ . Eigenspinors at different times can be related by

$$\Lambda(\tau_2) = L(\tau_2, \tau_1) \Lambda(\tau_1),$$

where the time-evolution operator is

$$L(\tau_2, \tau_1) = \Lambda(\tau_2) \bar{\Lambda}(\tau_1)$$

and is also seen to be a Lorentz rotation.

The time evolution is in principle found by solving the equation of motion

$$\dot{\Lambda} = \frac{1}{2} \mathbf{\Omega} \Lambda = \frac{1}{2} \Lambda \mathbf{\Omega}_r \quad (4.1)$$

with the spacetime rotation rate $\mathbf{\Omega} = \dot{\Lambda} \bar{\Lambda} - \Lambda \dot{\bar{\Lambda}} = \Lambda \mathbf{\Omega}_r \bar{\Lambda}$, where $\mathbf{\Omega}_r$ is its bivector value in the reference frame. This relation allows us to compute time-rates of change of any property that is known in the reference

frame. We take the reference frame of a massive particle to be the comoving inertial frame of the particle, in which $u = 1$. For example, the acceleration in the lab is

$$\dot{u} = \langle \mathbf{\Omega} u \rangle_{\Re} = \langle \Lambda \mathbf{\Omega}_r \Lambda^\dagger \rangle_{\Re} = \Lambda \langle \mathbf{\Omega}_r \rangle_{\Re} \Lambda^\dagger.$$

The proper velocity u of a particle can always be obtained from an eigenspinor that is a pure boost:

$$\Lambda = u^{1/2} = (1 + u) / \sqrt{2 \langle 1 + u \rangle_S}.$$

The spacetime rotation rate is then

$$\begin{aligned} \mathbf{\Omega} &= 2\dot{\Lambda}\bar{\Lambda} = \left\langle \left(\frac{d}{d\tau} \frac{1 + u}{\langle 1 + u \rangle_S^{1/2}} \right) \left(\frac{1 + \bar{u}}{\langle 1 + u \rangle_S^{1/2}} \right) \right\rangle_V \\ &= \frac{\langle \dot{u} (1 + \bar{u}) \rangle_V}{1 + \gamma} = \dot{\mathbf{u}} - \frac{\dot{\gamma} \mathbf{u}}{1 + \gamma} - i \frac{\dot{\mathbf{u}} \times \mathbf{u}}{1 + \gamma}, \end{aligned}$$

where we noted that $(1 + u)(1 + \bar{u})$ is a scalar. The negative imaginary part $\dot{\mathbf{u}} \times \mathbf{u} / (1 + \gamma)$ is the *spatial* rotation rate, known as the proper *Thomas precession rate*.⁹

4.2 Charge Dynamics in Uniform Fields

A standard problem in particle dynamics is to find the motion of a charge in constant, uniform electric and magnetic fields. We saw above an interpretation of the field \mathbf{F} as a spacetime plane. Its *definition* is given in this *operational* or dynamic sense: it is the spacetime rotation rate of a test charge with a unit charge-to-mass ratio. The Lorentz-force equation follows from the eigenspinor evolution (4.1) with the spacetime rotation rate $\mathbf{\Omega} = e\mathbf{F} / m$:

$$\dot{\Lambda} = \frac{e}{2m} \mathbf{F} \Lambda. \quad (4.2)$$

Exercise 53 From the relation $p = mu = m\Lambda\Lambda^\dagger$ for the momentum p of the charge, prove that the identification above of the spacetime rotation rate leads to the covariant Lorentz-force equation¹⁰

$$\dot{p} = e \langle \mathbf{F} u \rangle_{\Re} \equiv \frac{e}{2} (\mathbf{F} u + u \mathbf{F}^\dagger).$$

⁹This one-line derivation is not only much neater but considerably clearer than the usual cumbersome one based on differentials!

¹⁰One of the advantages of treating EM relativistically is that, provided we know how quantities transform, we can determine general laws from behavior in the rest frame. Thus the Lorentz force equation is the covariant extension of the *definition of the electric field*, viz. the force per unit charge in the rest frame of the charge: $\dot{p}_{\text{rest}} = e\mathbf{E}_{\text{rest}}$. This rest-frame relation is NOT covariant. The LHS is the rest-frame value of the covariant *paravector* \dot{p} , where the dot indicates differentiation with respect to proper

For any constant \mathbf{F} , the eigenspinor satisfying (4.2) has the form

$$\Lambda(\tau) = L(\tau) \Lambda(0), \quad L(\tau) = \exp\left(\frac{e}{2m} \mathbf{F} \tau\right)$$

and this implies a spacetime rotation of the proper velocity

$$u(\tau) = \Lambda(\tau) \Lambda^\dagger(\tau) = L(\tau) u(0) L^\dagger(\tau).$$

This is a trivial solution that works for all constant, uniform fields, whether or not they are spacelike, timelike, or null, simple or compound. Traditional texts usually treat the simple spacelike case (or occasionally the simple timelike case) by finding a drift frame in which the electromagnetic field is purely magnetic (or electric). We see that a more general solution is much easier. Furthermore, it is readily extended. If \mathbf{F} varies in time but commutes with itself at all different times, the solution has the form above with $\mathbf{F}\tau$ replaced by $\int_0^\tau \mathbf{F}(\tau') d\tau'$. Also, if you have a nonnull simple field, it can always be factored into

$$\mathbf{F} = u_d \mathbf{F}_d = u_d^{1/2} \mathbf{F}_d \bar{u}_d^{1/2}$$

where \mathbf{F}_d is the field in the *drift frame* and u_d is the proper velocity of the drift frame with respect to the lab. Its vector part is orthogonal to \mathbf{F}_d .

Example 54 *Consider the field*

$$\begin{aligned} \mathbf{F} &= (5\mathbf{e}_1 + 4i\mathbf{e}_3) E_0 = (5\mathbf{e}_1 + 4\mathbf{e}_1\mathbf{e}_2) E_0 \\ &= 5 \left(1 - \frac{4}{5}\mathbf{e}_2\right) E_0 \mathbf{e}_1 = u_d \mathbf{F}_d \end{aligned}$$

We note that $\mathbf{F}^2 > 0$, so we expect the field in the drift frame to be purely electric. We therefore factor out the electric field $E_0 \mathbf{e}_1$, leaving

$$\mathbf{F} = 5 \left(1 - \frac{4}{5}\mathbf{e}_2\right) E_0 \mathbf{e}_1 = u_d \mathbf{F}_d.$$

In the last step, we normalize the velocity factor so that u_d is a unit paravector:

$$u_d = \frac{\left(1 - \frac{4}{5}\mathbf{e}_2\right)}{\sqrt{1 - 16/25}} = \frac{5}{3} \left(1 - \frac{4}{5}\mathbf{e}_2\right) \equiv \gamma(1 + \mathbf{v})$$

which leaves $\mathbf{F}_d = 3E_0 \mathbf{e}_1$.

time, whereas the RHS is the *real part* of the covariant *biparavector* field \mathbf{F} , which transforms distinctly. The covariant Lorentz-force equation follows when we boost \dot{p} from rest to the lab:

$$\begin{aligned} \dot{p} &= \Lambda \dot{p}_{\text{rest}} \Lambda^\dagger = e \Lambda \langle \mathbf{F}_{\text{rest}} \rangle_{\mathfrak{R}} \Lambda^\dagger \\ &= e \langle \Lambda \mathbf{F}_{\text{rest}} \Lambda^\dagger \rangle_{\mathfrak{R}} = e \langle \Lambda (\bar{\Lambda} \mathbf{F} \Lambda) \Lambda^\dagger \rangle_{\mathfrak{R}} \\ &= e \langle \mathbf{F} u \rangle_{\mathfrak{R}}. \end{aligned}$$

Exercise 55 *Factor the electromagnetic field*

$$\mathbf{F} = (3\mathbf{e}_1 - 5i\mathbf{e}_3) E_0$$

into a drift velocity and electric or magnetic drift field.

Solution: $\mathbf{F} = u_d \mathbf{F}_d = \frac{5}{4} (1 + \frac{3}{5} \mathbf{e}_2) (-i4\mathbf{e}_3 E_0)$.

For *compound fields*, the drift field is a combination of collinear magnetic and electric fields, and the solution easily gives a *rifle transformation*: a commuting boost and rotation. Traditional electromagnetic-theory texts rarely treat this case. For *null fields*, \mathbf{F} with $\mathbf{F}^2 = 0$, the drift frame idea is not useful, since the drift velocity is at the speed of light. Our simple algebraic solution above still works, however, and indeed is then especially easy to evaluate since

$$\exp\left(\frac{1}{2}\boldsymbol{\Omega}\tau\right) = 1 + \frac{1}{2}\boldsymbol{\Omega}\tau$$

when $\boldsymbol{\Omega}^2 = 0$.

5 Maxwell's Equation

Maxwell's famous equations were written as a single quaternionic equation by Conway (1911)[12, 13], Silberstein (1912, 1914)[14, 15], and others. In APS we can write

$$\bar{\partial}\mathbf{F} = \mu_0 \bar{j}, \quad (5.1)$$

where $\mu_0 = \varepsilon_0^{-1} = 4\pi \times 30 \text{ Ohm}$ is the impedance of the vacuum, with $\dot{3} \equiv 2.99792458$. The usual four equations are simply the four vector grades of this relation, extracted as the real and imaginary, scalarlike and vectorlike parts. It is also seen as the *necessary covariant extension* of Coulomb's law $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$. The covariant field is not \mathbf{E} but $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, the divergence is part of the covariant gradient $\bar{\partial}$, and ρ must be part of $\bar{j} = \rho - \mathbf{j}$. The combination is Maxwell's equation.¹¹

Exercise 56 *Derive the continuity equation $\langle \bar{\partial}\bar{j} \rangle_S = 0$ in one step from Maxwell's equation. [Hint: note that the D'Alembertian $\partial\bar{\partial}$ is a scalar operator and that $\langle \mathbf{F} \rangle_S = 0$.]*

¹¹We have assumed that the source is a real paravector current and that there are no contributing pseudoparavector currents. Known currents are of the real paravector type, and a pseudoparavector current would behave counter-intuitively under parity inversion. Our assumption is supported experimentally by the apparent lack of magnetic monopoles.

5.1 Directed Plane Waves

In source-free space ($\bar{j} = 0$), there are solutions $\mathbf{F}(s)$ that depend on spacetime position only through the Lorentz invariant $s = \langle k\bar{x} \rangle_0 = \omega t - \mathbf{k} \cdot \mathbf{x}$, where $k = \omega + \mathbf{k} \neq 0$ is a constant propagation paravector. Since $\partial \langle k\bar{x} \rangle_0 = k$, Maxwell's equation gives

$$\bar{\partial}\mathbf{F} = \bar{k}\mathbf{F}'(s) = 0. \quad (5.2)$$

In a division algebra, we could divide by \bar{k} and conclude that $\mathbf{F}'(s) = 0$, a rather uninteresting solution. There is another possibility here because APS is not a division algebra: \bar{k} may have no inverse. Then k has the form $k = \omega(1 + \hat{\mathbf{k}})$, and after integrating (5.2) from some s_0 at which \mathbf{F} is presumed to vanish, we get $(1 - \hat{\mathbf{k}})\mathbf{F}(s) = 0$, which means

$$\mathbf{F}(s) = \hat{\mathbf{k}}\mathbf{F}(s).$$

The scalar part of \mathbf{F} vanishes and consequently $\langle \hat{\mathbf{k}}\mathbf{F}(s) \rangle_S = \hat{\mathbf{k}} \cdot \mathbf{F}(s) = 0$ so that the fields \mathbf{E} and \mathbf{B} are perpendicular to $\hat{\mathbf{k}}$ and thus anticommute with it. Furthermore, equating imaginary parts gives

$$i\mathbf{B} = \hat{\mathbf{k}}\mathbf{E}$$

and it follows that

$$\begin{aligned} \mathbf{F} &= \mathbf{E} + i\mathbf{B} \\ &= (1 + \hat{\mathbf{k}})\mathbf{E}(s) \end{aligned}$$

with $\mathbf{E} = \langle \mathbf{F} \rangle_1$ real. This is a plane-wave solution with \mathbf{F} constant on spatial planes perpendicular to $\hat{\mathbf{k}}$. Such planes propagate at the speed of light along $\hat{\mathbf{k}}$. In spacetime, \mathbf{F} is constant on the lightcone $\hat{\mathbf{k}} \cdot \mathbf{x} = t$. However, \mathbf{F} is not necessarily monochromatic, since $\mathbf{E}(s)$ can have any

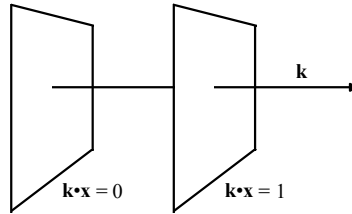


FIGURE 8. Constant spatial planes of a directed plane wave propagate at ω/\mathbf{k} .

functional form, including a pulse, and the scale factor ω , although it

has dimensions of frequency, may have nothing to do with any physical oscillation. Note further that \mathbf{F} is *null*.¹²

$$\mathbf{F}^2 = (1 + \hat{\mathbf{k}}) \mathbf{E} (1 + \hat{\mathbf{k}}) \mathbf{E} = (1 + \hat{\mathbf{k}}) (1 - \hat{\mathbf{k}}) \mathbf{E}^2 = 0 .$$

The energy density $\mathcal{E} = \frac{1}{2} (\varepsilon_0 \mathbf{E}^2 + \mathbf{B}^2 / \mu_0)$ and Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0$ are given by

$$\frac{1}{2} \varepsilon_0 \mathbf{F} \mathbf{F}^\dagger = \mathcal{E} + \mathbf{S} = \varepsilon_0 \mathbf{E}^2 (1 + \hat{\mathbf{k}}) .$$

Example 57 *Monochromatic plane wave of frequency ω linearly polarized along \mathbf{E}_0 :*

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \cos s$$

Example 58 *Monochromatic plane wave of frequency 5ω linearly polarized along \mathbf{E}_0 :*

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \cos 5s$$

Example 59 *Monochromatic plane wave circularly polarized with helicity κ :*

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp(i\kappa s \hat{\mathbf{k}}) = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp(-i\kappa s)$$

Note that the rotation factor has become a phase factor (a “duality rotation”) in the last expression. This is a result of the “Pacwoman property” in which $1 + \hat{\mathbf{k}}$ gobbles neighboring factors of $\hat{\mathbf{k}}$: $(1 + \hat{\mathbf{k}}) \hat{\mathbf{k}} = (1 + \hat{\mathbf{k}})$:

$$\begin{aligned} (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp(i\kappa s \hat{\mathbf{k}}) &= (1 + \hat{\mathbf{k}}) \exp(-i\kappa s \hat{\mathbf{k}}) \mathbf{E}_0 \\ &= (1 + \hat{\mathbf{k}}) (\cos \kappa s - i \hat{\mathbf{k}} \sin \kappa s) \mathbf{E}_0 \\ \text{[gobble!]} &= (1 + \hat{\mathbf{k}}) (\cos \kappa s - i \sin \kappa s) \mathbf{E}_0 \\ &= (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp(-i\kappa s) . \end{aligned}$$

Example 60 *Linearly polarized gaussian pulse of width Δ/ω :*

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp\left(-\frac{1}{2} s^2 / \Delta^2\right)$$

Example 61 *A circularly polarized gaussian pulse with center frequency ω :*

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp\left(-\frac{1}{2} s^2 / \Delta^2 + i s\right)$$

¹²In fact, \mathbf{F} is what Penrose calls a *null flag*. The *flagpole* $(1 + \hat{\mathbf{k}})$ lies in the plane of the flag but is orthogonal to it. This becomes important below when we discuss charge dynamics. The null flag structure is beautiful and powerful, but you miss it entirely if you write only separate electric and magnetic fields!

These all have the common form

$$\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E} = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 f(s)$$

where $f(s)$ is a scalar function, possibly complex valued. We will use this below.¹³

5.2 Polarization Basis

A beam of monochromatic radiation can be elliptically polarized as well as linearly or circularly polarized. There are two degrees of freedom, so that arbitrary polarization can be expressed as a linear combination of two independent polarization types. There is a close analogy to the 2-D oscillations of a pendulum formed by hanging a mass on a string. Both linear and circular polarization bases are common, but we find the circular basis most convenient, partially because of the relation noted above between spatial and duality rotations. Circularly polarized waves also have the simple form used popularly by R. P. Feynman[16] in terms of the paravector potential as a rotating real vector:

$$A_\kappa = \mathbf{a} \exp(i\kappa s \hat{\mathbf{k}}),$$

with $s = \langle k\bar{x} \rangle_S = \omega t - \mathbf{k} \cdot \mathbf{x}$ and $\mathbf{a} \cdot \mathbf{k} = 0$, where $\kappa = \pm 1$ is the helicity. The corresponding field is

$$\mathbf{F}_\kappa = \langle \partial \bar{A} \rangle_V = i\kappa k \mathbf{a} \exp(i\kappa s \hat{\mathbf{k}}) = (1 + \hat{\mathbf{k}}) \mathbf{E}_0 \exp(i\kappa s \hat{\mathbf{k}}),$$

with $\mathbf{E}_0 = i\kappa k \mathbf{a} = \kappa \mathbf{a} \times \mathbf{k}$.

A linear combination of both helicities of such directed waves is given by

$$\begin{aligned} \mathbf{F} &= (1 + \hat{\mathbf{k}}) \hat{\mathbf{E}}_0 e^{i\delta \hat{\mathbf{k}}} (E_+ e^{is \hat{\mathbf{k}}} + E_- e^{-is \hat{\mathbf{k}}}) \\ &= (1 + \hat{\mathbf{k}}) \hat{\mathbf{E}}_0 e^{-i\delta} (E_+ e^{-is} + E_- e^{is}), \end{aligned}$$

where E_\pm are the real field amplitudes, δ gives the rotation of \mathbf{E} about $\hat{\mathbf{k}}$ at $s = 0$, and in the second line, we let Pacwoman gobble the $\hat{\mathbf{k}}$'s. Because every directed plane wave can be expressed in the form $\mathbf{F} = (1 + \hat{\mathbf{k}}) \mathbf{E}(s)$,

¹³Warning: Don't assume from the last relation that $\mathbf{E}(s)$ is $\mathbf{E}_0 f(s)$. It doesn't follow when f is complex. Remember that $(1 + \hat{\mathbf{k}})$ has no inverse, so we can't simply drop it. Instead, since \mathbf{E}_0 is real and $\hat{\mathbf{k}} \mathbf{E}_0$ is imaginary,

$$\mathbf{E} = \left\langle (1 + \hat{\mathbf{k}}) \mathbf{E}_0 f(s) \right\rangle_{\Re} = \mathbf{E}_0 \langle f \rangle_{\Re} + \hat{\mathbf{k}} \mathbf{E}_0 \langle f \rangle_{\Im}.$$

it is sufficient to determine $\mathbf{E}(s) = \langle \mathbf{F} \rangle_{\Re}$:

$$\begin{aligned} \mathbf{E} &= \left\langle \left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{E}}_0 E_+ e^{-i\delta} e^{-is} + \left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{E}}_0 E_- e^{-i\delta} e^{is} \right\rangle_{\Re} \\ &= \left\langle \left[\left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{E}}_0 E_+ e^{-i\delta} + \hat{\mathbf{E}}_0 \left(1 + \hat{\mathbf{k}}\right) E_- e^{i\delta} \right] e^{-is} \right\rangle_{\Re} \\ &= \langle (\boldsymbol{\epsilon}_+, \boldsymbol{\epsilon}_-) \Phi e^{-is} \rangle_{\Re} , \end{aligned}$$

where the complex polarization basis vectors $\boldsymbol{\epsilon}_{\pm} = 2^{-1/2} (1 \pm \hat{\mathbf{k}}) \hat{\mathbf{E}}_0$ are null flags satisfying $\boldsymbol{\epsilon}_- = \boldsymbol{\epsilon}_+^{\dagger}$, $\boldsymbol{\epsilon}_+ \cdot \boldsymbol{\epsilon}_+^{\dagger} = 1 = \boldsymbol{\epsilon}_- \cdot \boldsymbol{\epsilon}_-^{\dagger}$, and the *Poincaré spinor*

$$\Phi = \sqrt{2} \begin{pmatrix} E_+ e^{-i\delta} \\ E_- e^{i\delta} \end{pmatrix}$$

gives the (real) electric-field amplitudes and their phases, and thus contains all the information needed to determine the polarization and intensity of the wave.¹⁴

Stokes Parameters

Physical beams of radiation are not fully monochromatic and not necessarily fully polarized. To describe partially polarized light, we can use the *coherency density*, which in the case of a single Poincaré spinor is defined by

$$\rho = \varepsilon_0 \Phi \Phi^{\dagger} = \rho^{\mu} \boldsymbol{\sigma}_{\mu} ,$$

where the $\boldsymbol{\sigma}_{\mu}$ are the usual Pauli spin matrices and the normalization factor ε_0 has been chosen to make ρ^0 the time-averaged energy density:

$$\begin{aligned} \langle \mathcal{E} + \mathbf{S} \rangle_{t\text{-av}} &= \left\langle \frac{\varepsilon_0}{2} \mathbf{F} \mathbf{F}^{\dagger} \right\rangle_{t\text{-av}} = \frac{\varepsilon_0}{2} \left\langle \left(1 + \hat{\mathbf{k}}\right) \mathbf{E}^2 \left(1 + \hat{\mathbf{k}}\right) \right\rangle_{t\text{-av}} \\ &= \varepsilon_0 \left(1 + \hat{\mathbf{k}}\right) \langle \mathbf{E}^2 \rangle_{t\text{-av}} = \rho^0 \left(1 + \hat{\mathbf{k}}\right) . \end{aligned}$$

¹⁴The direction of the magnetic field at $s=0$ is $\hat{\mathbf{B}}_0 = \hat{\mathbf{k}} \times \hat{\mathbf{E}}_0$. In terms of it

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}} \left(\hat{\mathbf{E}}_0 \pm i \hat{\mathbf{B}}_0 \right) .$$

The electric field \mathbf{E} can be transformed to the familiar *Jones-vector* basis by a unitary matrix:

$$\begin{aligned} \mathbf{E} &= \left\langle \left(\hat{\mathbf{E}}_0, \hat{\mathbf{B}}_0 \right) \Phi_J e^{-is} \right\rangle_{\Re} \\ \Phi_J &= U_J \Phi = \begin{pmatrix} E_+ e^{-i\delta} + E_- e^{i\delta} \\ i(E_+ e^{-i\delta} - E_- e^{i\delta}) \end{pmatrix} \\ \left(\hat{\mathbf{E}}_0, \hat{\mathbf{B}}_0 \right) &= (\boldsymbol{\epsilon}_+, \boldsymbol{\epsilon}_-) U_J^{\dagger} \\ U_J &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} . \end{aligned}$$

The coefficients ρ^μ are the *Stokes parameters*. The coherency density can be treated algebraically in \mathcal{Cl}_3 to study all polarization and intensity properties of the beam.¹⁵

The Stokes parameters are given by

$$\rho^\mu = \langle \rho \sigma_\mu \rangle_S = \frac{1}{2} \text{tr}(\rho \sigma_\mu).$$

Explicitly

$$\begin{aligned} \rho^0 &= \varepsilon_0 (E_+^2 + E_-^2) \\ \rho^1 &= 2\varepsilon_0 E_+ E_- \cos \phi \\ \rho^2 &= 2\varepsilon_0 E_+ E_- \sin \phi \\ \rho^3 &= \varepsilon_0 (E_+^2 - E_-^2), \end{aligned}$$

where $\phi = 2\delta$ is the azimuthal angle of $\boldsymbol{\rho} = \rho^1 \boldsymbol{\sigma}_1 + \rho^2 \boldsymbol{\sigma}_2 + \rho^3 \boldsymbol{\sigma}_3$. The coherency density is a paravector in the space spanned by the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$

$$\rho = \rho^0 + \boldsymbol{\rho}.$$

This space, called *Stokes subspace*, is a 3-D Euclidean space analogous to physical space. It is not physical space, but its geometric algebra has exactly the same form as (is isomorphic to) APS, and it illustrates how Clifford algebras can arise in physics for spaces other than physical space. As in APS, it is the algebra and not the explicit matrix representation that is significant.

As defined for a single Φ , ρ is null: $\det \rho = \rho \bar{\rho} = 0$. Thus,

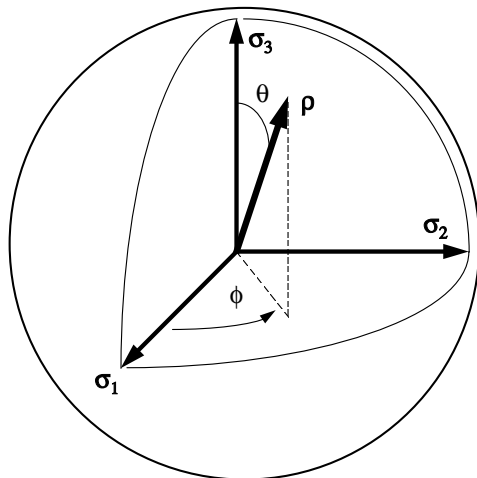
$$\rho = \rho^0 (1 + \mathbf{n})$$

where \mathbf{n} is a unit vector in the direction of $\boldsymbol{\rho}$. It fully specifies the type of polarization. In particular, for positive helicity light, $\mathbf{n} = \boldsymbol{\sigma}_3$, for negative helicity polarization $\mathbf{n} = -\boldsymbol{\sigma}_3$, and for linear polarization at an angle $\delta = \phi/2$ with respect to \mathbf{E}_0 , $\mathbf{n} = \boldsymbol{\sigma}_1 \cos \phi + \boldsymbol{\sigma}_2 \sin \phi$. Other directions correspond to elliptical polarization.

Polarizers and Phase Shifters

The action of ideal polarizers and phase shifters on the wave is modeled mathematically by transformations on the Poincaré spinor Φ of the form

¹⁵Many optics texts still use the 4×4 Mueller matrices for this purpose, but this strikes me as even more perverse than using 4×4 matrices for Lorentz transformations. The coherency density, introduced by Born and Wolf as the “coherency matrix” by the time of the third edition of their **Principles of Optics** book in 1964, is really much simpler. Transformed as here into the helicity basis, it matches the quantum formulation of the spin-1/2 density matrix as well as the standard matrix representation of APS.

FIGURE 9. The direction of ρ gives the type of polarization.

$\Phi \rightarrow T\Phi$. For *polarizers* T is a *projector*

$$P_{\mathbf{n}} = \frac{1}{2}(1 + \mathbf{n}),$$

where \mathbf{n} is a real unit vector in Stokes subspace that specifies the type of polarization. Projectors are real idempotent elements: $P_{\mathbf{n}} = P_{\mathbf{n}}^\dagger = P_{\mathbf{n}}^2$, just as we would expect for ideal polarizers. For example, a circular polarizer allowing only waves of positive helicity corresponds to the projector $P_{\sigma_3} = \frac{1}{2}(1 + \sigma_3)$, which when applied to Φ eliminates the contribution E_- of negative helicity without affecting the positive-helicity part:

$$\Phi := \sqrt{2} \begin{pmatrix} E_+ e^{-i\delta} \\ E_- e^{i\delta} \end{pmatrix} \rightarrow P_{\sigma_3} \Phi := \sqrt{2} \begin{pmatrix} E_+ e^{-i\delta} \\ 0 \end{pmatrix}.$$

A second application of P_{σ_3} changes nothing further. The polarizer represented by the complementary projector \bar{P}_{σ_3} eliminates the upper component of Φ . Generally, since $\bar{P}_{\mathbf{n}} P_{\mathbf{n}} = P_{-\mathbf{n}} P_{\mathbf{n}} = 0$, opposite directions in Stokes subspace correspond to orthogonal polarizations.

Multiplication of Φ by $\exp(i\alpha)$ phase shifts the wave by an angle α . An overall phase shift in the wave is hardly noticeable since the total phase is in any case changing very rapidly, but the effect of giving different polarization components different shifts can be important. If the wave is split into orthogonal polarization components ($\pm \mathbf{n}$) and the two components are given a relative shift of α , the result is equivalent to rotating ρ by α about \mathbf{n} in Stokes subspace:

$$T = P_{\mathbf{n}} e^{i\alpha/2} + \bar{P}_{\mathbf{n}} e^{-i\alpha/2} = e^{i\alpha/2}.$$

If $\mathbf{n} = \boldsymbol{\sigma}_3$, this operator represents the effects of passing the waves through a medium with different indices of refraction for circularly polarized light of different helicities, as in the Faraday effect or in optically active organic solutions, and the result is a rotation of the plane of linear polarization by $\alpha/2$ about \mathbf{k} . On the other hand, if \mathbf{n} lies in the $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ plane, the above operator T represents the effect of a birefringent medium with polarization types \mathbf{n} and $-\mathbf{n}$ corresponding to the slow and fast axes, respectively. In a quarter-wave plate, for example, $\alpha = \pi/2$. Incident light linearly polarized half way between the fast and slow axes will be rotated by $\pi/2$ to $\pm \boldsymbol{\sigma}_3$, giving circularly polarized light.

The basic technique of splitting the light into opposite polarizations $\pm \mathbf{n}$, acting differently on the two polarization components, and then recombining is modeled by the operator

$$T = P_{\mathbf{n}} A_+ + P_{-\mathbf{n}} A_- .$$

If the actions A_{\pm} are the same, the result is the identity operator: nothing happens. The ideal filter is the special case $A_+ = 1$, and $A_- = 0$, that is one of the polarization parts is discarded.

Coherent Superpositions and Incoherent Mixtures

A superposition of two waves of the same frequency is *coherent* because their relative phase is fixed. Mathematically, one adds spinors in such cases:

$$\Phi = \Phi_1 + \Phi_2 ,$$

where the subscripts refer to the two waves, not to spinor components. Coherent superpositions of monochromatic waves are always fully polarized and can be represented by a single Poincaré spinor or Jones vector.

However, real beams of waves are never fully monochromatic. Two waves of different frequencies have a continually changing relative phase, and when their product $\Phi_1 \Phi_2^\dagger$ is averaged over periods large relative to their beat period, such terms vanish. The waves then combine *incoherently*, and one should add their coherency densities rather than their Poincaré spinors:

$$\rho = \rho_1 + \rho_2 .$$

In such an incoherent superposition, the polarization can vary from 0 to 100%.

Any transformation T of spinors, $\Phi \rightarrow T\Phi$, transforms the coherency density by

$$\rho \rightarrow T\rho T^\dagger .$$

Example 62 Consider a sandwich of two crossed linear polarizers, with a third linear polarizer of intermediate polarization inserted in between. If ρ is initially unpolarized, the first polarizer, say of type \mathbf{n} , produces

$$P_{\mathbf{n}} \rho^0 P_{\mathbf{n}} = \rho^0 P_{\mathbf{n}} = \frac{1}{2} \rho^0 (1 + \mathbf{n}) ,$$

that is, fully polarized light with half the intensity. The crossed polarizer, represented by $P_{-\mathbf{n}}$, would annihilate the polarized beam, but if a different polarizer, say type \mathbf{m} , is applied before the $-\mathbf{n}$ one, we get

$$\begin{aligned} P_{\mathbf{m}}\rho^0 P_{\mathbf{n}} P_{\mathbf{m}} &= \rho^0 (P_{\mathbf{m}} P_{\mathbf{n}} + \bar{P}_{\mathbf{n}} \bar{P}_{\mathbf{m}}) P_{\mathbf{m}} \\ &= 2\rho^0 \langle P_{\mathbf{m}} P_{\mathbf{n}} \rangle_S P_{\mathbf{m}} \\ &= \rho^0 \frac{(1 + \mathbf{m} \cdot \mathbf{n})}{2} P_{\mathbf{m}} \end{aligned}$$

Application of the final polarizer of type $-\mathbf{n}$ then yields

$$\begin{aligned} \rho^0 \frac{(1 + \mathbf{m} \cdot \mathbf{n})}{2} P_{-\mathbf{n}} P_{\mathbf{m}} P_{-\mathbf{n}} &= \rho^0 \frac{(1 + \mathbf{m} \cdot \mathbf{n})}{2} \frac{(1 - \mathbf{m} \cdot \mathbf{n})}{2} P_{-\mathbf{n}} \\ &= \rho^0 \frac{(1 - (\mathbf{m} \cdot \mathbf{n})^2)}{4} P_{-\mathbf{n}} . \end{aligned}$$

The maximum intensity is 1/8th the initial, reached when $\mathbf{m} \cdot \mathbf{n} = 0$, that is when the linear polarization of the intervening polarizer is at 45° to each of the crossed polarizers.

In addition, many other transformations that do not preserve the polarization can be applied. For example, *depolarization* of a fraction f of the radiation is modeled by

$$\rho \rightarrow (1 - f)\rho + f \langle \rho \rangle_S ,$$

and detection itself takes the form

$$\rho \rightarrow \langle \rho D \rangle_S ,$$

where the detection operator may equal $D = 1$ in an ideal case, but more generally it can have different efficiencies D_{\pm} for opposite polarization types:

$$D = D_+ P_{\mathbf{n}} + D_- P_{-\mathbf{n}} .$$

A number of other transformations are possible.

5.3 Standing Waves and $\mathbf{E} \parallel \mathbf{B}$ Fields

Standing waves are formed from the superposition of oppositely directed plane waves:

$$\begin{aligned} \mathbf{F} &= (1 + \hat{\mathbf{k}}) \mathbf{E}_0 f(\omega t - \mathbf{k} \cdot \mathbf{x}) + (1 - \hat{\mathbf{k}}) \mathbf{E}_0 f(\omega t + \mathbf{k} \cdot \mathbf{x}) \\ &= (f_+ - f_- \hat{\mathbf{k}}) \mathbf{E}_0 \end{aligned}$$

where $f_{\pm} = f(\omega t + \mathbf{k} \cdot \mathbf{x}) \pm f(\omega t - \mathbf{k} \cdot \mathbf{x})$. For the monochromatic circularly polarized wave $f(s) = \exp is$ of negative helicity,

$$\begin{aligned} f_+ &= 2 \exp(i\omega t) \cos \mathbf{k} \cdot \mathbf{x} \\ f_- &= 2i \exp(i\omega t) \sin \mathbf{k} \cdot \mathbf{x} \end{aligned}$$

and

$$\begin{aligned} \mathbf{F} &= 2 \left(\cos \mathbf{k} \cdot \mathbf{x} - i \hat{\mathbf{k}} \sin \mathbf{k} \cdot \mathbf{x} \right) \mathbf{E}_0 \exp(i\omega t) \\ &= \underbrace{2\mathbf{E}_0 \exp(i\hat{\mathbf{k}} \mathbf{k} \cdot \mathbf{x})}_{\text{real: spatial rot.}} \underbrace{\exp(i\omega t)}_{\text{duality rot.}}, \end{aligned}$$

which represents electric and magnetic fields that are aligned on the radii of a spiral fixed in space. The field rotates in duality space ($\mathbf{E} \rightarrow \mathbf{B} \rightarrow -\mathbf{E}$) at every point in space. Thus at $t = 0$, there is only an electric field throughout space, and a quarter of a cycle later, there is only a magnetic field. At intermediate times, both electric and magnetic fields exist and they are aligned. To change to a wave of positive helicity plus its reflection, replace i by $-i$.

The roles of time and space are reversed if waves of opposite helicity are superimposed. In this case, the fields throughout space point in a single direction at a given instant in time, and that direction rotates about $\hat{\mathbf{k}}$ in time. How much of the field is electric and how much magnetic depends only on the spatial position along $\hat{\mathbf{k}}$. The energy density for both circular-wave superpositions are constant and their Poynting vectors vanish: $\mathcal{E} + \mathbf{S} = \frac{1}{2}\varepsilon_0 \mathbf{F}\mathbf{F}^\dagger = 2\varepsilon_0 \mathbf{E}_0^2$.¹⁶

5.4 Charge Dynamics in Plane Waves

We saw above how eigenspinors made easy work of the problem of charge dynamics in constant uniform fields. What about motion in the more complicated field of a directed plane wave? A derivation of the relativistic motion of a charge in a linearly polarized monochromatic plane wave was given by A. H. Taub[17] in 1948, using 4×4 matrix Lorentz transformations of

¹⁶There was considerable controversy about such fields when they were first proposed by Chu and Ohkawa in 1982 [C. Chu and T. Ohkawa, *Phys. Rev. Lett.* **48** (1982) 837]. A surprising number of physicists were convinced that the rule $\mathbf{E} \cdot \mathbf{B} = 0$ for directed plane waves and other simple fields also had to be true of superpositions in free space. The result and its interpretation are quite obvious with some geometric algebra. Now $\mathbf{E} \parallel \mathbf{B}$ fields are used routinely in the laser cooling of atoms to submicrokelvin temperatures. Opposite helicities, as naturally obtained by reflection, are required for the operation of the *Sisyphus effect*.

the charge. The derivation is simpler using eigenspinors (or rotors) in Clifford's geometric algebra, as shown by Hestenes in 1974[18]. Here we review the extension[6] to arbitrary plane waves and plane-wave pulses in APS.

The fields of directed plane waves are null flags of the form

$$\mathbf{F}(s) = (1 + \hat{\mathbf{k}}) \mathbf{E}(s), \quad s = \langle k\bar{x} \rangle_S, \quad \hat{\mathbf{k}} \cdot \mathbf{E} = \mathbf{0}.$$

All null flags on a given flagpole annihilate each other and therefore commute:

$$\mathbf{F}(s_1) \mathbf{F}(s_2) = (1 + \hat{\mathbf{k}}) \mathbf{E}(s_1) (1 + \hat{\mathbf{k}}) \mathbf{E}(s_2) = 0 = \mathbf{F}(s_2) \mathbf{F}(s_1)$$

and as a result, the solution to the equation of motion (4.2) is the exponential expression

$$\Lambda(\tau) = \exp \left\{ \frac{e}{2m} \int_0^\tau \mathbf{F}[s(\tau')] d\tau' \right\} \Lambda(0),$$

which by virtue of the nilpotency of \mathbf{F} reduces to

$$\Lambda(\tau) = \left\{ 1 + \frac{e}{2m} \int_0^\tau \mathbf{F}[s(\tau')] d\tau' \right\} \Lambda(0). \quad (5.3)$$

The problem is that $\mathbf{F}[s(\tau)]$ is the electromagnetic field at the charge at proper time τ , and we need to know the world line $x(\tau)$ of the charge to know its value. We could get the world line by integrating $u(\tau) = \Lambda\Lambda^\dagger$, but it's Λ we're trying to find! This looks hopeless, but there's a surprising symmetry that comes to the rescue.

From $\bar{k}\mathbf{F} = 0$, it follows that $\bar{k}\dot{\Lambda}(\tau) = 0$, the bar conjugate of which is $\bar{\Lambda}k = 0$, and from the definition of Λ , $\bar{\Lambda}k\bar{\Lambda}^\dagger = k_{\text{rest}}$ is the propagation paravector as seen in the instantaneous rest frame of the charge. Since k is constant in the lab,

$$\dot{k}_{\text{rest}} = \frac{d}{d\tau} (\bar{\Lambda}k\bar{\Lambda}^\dagger) = 2 \left\langle \frac{d}{d\tau} (\bar{\Lambda}k\bar{\Lambda}^\dagger) \right\rangle_{\mathbb{R}} = 0,$$

and it is also constant in the rest frame of the charge. Now that is unexpected because the charge, as we will see, is *accelerating*! In particular, the scalar part of k_{rest} ,

$$\dot{s} = \langle k\bar{u} \rangle_S = \omega_{\text{rest}},$$

is constant. Thus, $s = s_0 + \omega_{\text{rest}}\tau$ and we can solve (5.3) above:

$$\begin{aligned} \Lambda(\tau) &= \left[1 + \frac{e}{2m\omega_{\text{rest}}} \int_{s_0}^s \mathbf{F}(s') ds' \right] \Lambda(0) \\ &= \left[1 + \frac{e(1 + \hat{\mathbf{k}}) \mathbf{E}_0}{2m\omega_{\text{rest}}} \int_{s_0}^s f(s') ds' \right] \Lambda(0), \end{aligned}$$

where in the second line we noted that every plane wave with flag pole $1 + \hat{\mathbf{k}}$ can be expressed by $(1 + \hat{\mathbf{k}}) \mathbf{E}_0 f(s)$. This is the solution since we know $f(s)$ and can integrate it, but there's another form, using

$$\mathbf{F}(s) = \partial \bar{A}(s) = k \bar{A}'(s) ,$$

which is valid in the Lorenz gauge $\langle \partial \bar{A} \rangle_S = 0$. The integral over \mathbf{F} is thus trivially expressed as

$$\Lambda(\tau) - \Lambda(0) = \frac{ek}{2m\omega_{\text{rest}}} [\bar{A}(s) - \bar{A}(s_0)] \Lambda(0) , \quad (5.4)$$

giving a change in the eigenspinor that is linear in the change in the paravector potential. This can lead to substantial accelerations in the plane of \mathbf{k} and $\mathbf{A}(s) - \mathbf{A}(s_0)$, especially when the charge is injected with high velocity along \mathbf{k} into the beam so as to produce a large Doppler shift and thus a large ratio $\omega/\omega_{\text{rest}}$. Curiously, however, the acceleration occurs always in a way that conserves $\langle k\bar{u} \rangle_S$. There can therefore be substantial first- and even second-order Doppler shifts from the acceleration caused by the field, but the *total* Doppler shift in the frame of the charge is constant.

Example 63 *For the field pulse*

$$\mathbf{F}(s) = k\mathbf{A}_0 / \cosh^2 s$$

there is a net change in the paravector potential of $-2\mathbf{A}_0$ so that from (5.4),

$$\Lambda(\infty) = \left(1 + \frac{k\mathbf{a}_0}{\omega_{\text{rest}}}\right) \Lambda(0) .$$

With injection of the charge along $\hat{\mathbf{k}}$, one finds a final proper velocity

$$u(\infty) = \Lambda(\infty) \Lambda^\dagger(\infty) = u_0 + 2\mathbf{a}_0 + \frac{2k\mathbf{a}_0^2}{\omega_{\text{rest}}}$$

where $\mathbf{a}_0 = e\mathbf{A}_0/m$ is the dimensionless amplitude of the vector potential.

Exercise 64 *Show that the energy gain in the last example is*

$$\frac{2\omega\mathbf{a}_0^2}{\omega_{\text{rest}}} m .$$

Exercise 65 *Verify that $u\bar{u}$ is conserved in the last example. [Hint: recall $\langle k\bar{u}_0 \rangle_S = \omega_{\text{rest}}$ and note $u_0\mathbf{a}_0 = \mathbf{a}_0\bar{u}_0$ and $k\mathbf{a}_0 = \mathbf{a}_0\bar{k}$.]*

Insight into why k_{rest} stays constant in the frame of the accelerating charge is provided by the geometry of the null flag. The rotation occurs in the flag plane, and the propagation paravector lies along the flagpole which

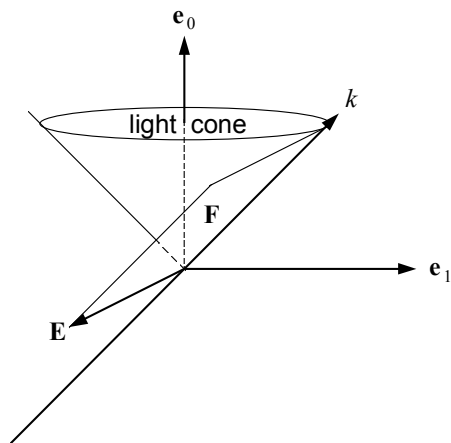


FIGURE 10. The electromagnetic field of a directed plane wave is a flag tangent to the light cone. Its flagpole lies along the propagation paravector k .

lies in the plane. However, the flagpole is also *orthogonal* to the plane and therefore invariant under rotations in it.

Modern lasers possess high electric fields and seem excellent candidates for particle accelerators. Our result (5.4) above shows why simply hitting a charge with a laser has rather limited effect. The net change in the eigenspinor is linear in the change in paravector potential. You can gain energy only for about a half cycle of the laser. Continue into the next half cycle and the charge starts losing energy.

Another scheme is possible that avoids these problems. It is called the autoresonant laser accelerator (ALA) and combines a constant magnetic field along the $\hat{\mathbf{k}}$ axis of a circularly polarized plane wave. When the cyclotron frequency of the charge is resonant with the frequency of the laser, the acceleration along $\hat{\mathbf{k}}$ is continuous and the frequencies remain resonant. This can lead to substantial energy gains. Eigenspinors and projectors provide a powerful tool for solving trajectories of charges in such cases, as well as for cases of superimposed electric fields[19].

The electromagnetic field in the case of a longitudinal magnetic field has the form

$$\mathbf{F} = k\bar{A}'(s) + iB_0\hat{\mathbf{k}}$$

from which the equation of motion times \bar{k} follows:

$$\bar{k}\dot{\Lambda} = \frac{e}{2m}\bar{k}\mathbf{F}\Lambda = -\frac{i\omega_c}{2}\bar{k}\Lambda ,$$

where $\omega_c = eB_0/m$ is the proper cyclotron frequency. The solution

$$\bar{k}\Lambda = \exp(-i\omega_c\tau/2)\bar{k}\Lambda(0)$$

implies

$$\dot{s} = \langle \Lambda^\dagger \bar{k} \Lambda \rangle_S = \omega_{\text{rest}} = \text{const.}$$

as before, so that we can again solve for $\dot{\Lambda}$. At resonance $\omega_c = \omega_{\text{rest}}$ in a circularly polarized wave $A = m \exp(i(s - s_0)\hat{\mathbf{k}}) \mathbf{a}/e$, we find energy gains of $m\Delta\gamma$ with

$$\Delta\gamma = \mathbf{u}(0) \cdot \mathbf{k} \times \mathbf{a}\tau + \frac{1}{2}\omega\omega_c(\mathbf{a}\tau)^2.$$

These can be substantial, accelerating 100 MeV electrons to 1 TeV within 2 km in a 10 Tesla magnetic field with a Ti:Sapphire laser pulse.[19]

5.5 Potential of Moving Point Charge

Accelerating charges radiate. Indeed, radiation reaction had to be calculated for the ALA, even though it turned out to be small over a wide range of parameters. The potentials and fields of point charges in general motion are derived in most electrodynamics texts. The traditional derivation of the retarded fields is tricky and the final result rather messy and opaque. APS helps clear the fog and provides insight into the origin and nature of the radiation.

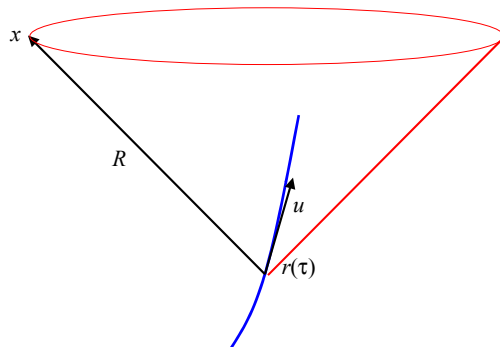


FIGURE 11. x is field point, $r(\tau)$ is world line of charge.

The potential of a moving point charge was derived (before special relativity!) by A. Liénard (1898) and E. Wiechert (1900). It is easily obtained from the rest-frame Coulomb potential:

$$\Phi_{\text{rest}} = \frac{K_0 e}{\langle R_{\text{rest}} \rangle_S}$$

with $K_0 = (4\pi\epsilon_0)^{-1}$ and $R(\tau) = x - r(\tau)$, the difference paravector between the field position x and the world line of the charge at the retarded

proper time τ . The denominator $\langle R_{\text{rest}} \rangle_S$ is the time component of the difference in the commoving inertial frame. It is a Lorentz invariant simply because it is measured in the rest frame of the charge at the retarded time, but we make it manifestly invariant by writing

$$\langle R_{\text{rest}} \rangle_S = \langle R\bar{u} \rangle_S ,$$

where u is the proper velocity of the charge at τ and $R = R(\tau)$ can be in any inertial frame. We only need to boost Φ_{rest} to the proper velocity of the charge at the retarded τ to get the paravector potential for a charge in general motion:

$$A(x) = \Lambda \Phi_{\text{rest}} \Lambda^\dagger = \frac{K_0 e u}{\langle R\bar{u} \rangle_S} .$$

It is that simple. Note that $A(x)$ is covariant and depends only on the relative position and proper velocity of the charge at the retarded τ . It is independent of the acceleration or any other properties of the trajectory. The retarded time τ is determined by the light-cone condition $R\bar{R} = 0$, where causality demands that $R^0 > 0$.

Liénard-Wiechert Field

The electromagnetic field is found from $\mathbf{F} = \langle \partial \bar{A} \rangle_V$ with the complication that the light-cone condition makes τ a function of x , that is a scalar field, so that there are two contributions to the gradient term:

$$\partial = (\partial)_\tau + [\partial\tau(x)] \frac{d}{d\tau} .$$

The first term is differentiation with respect to the field position x with τ held fixed. The second term arises from the dependence on x of the scalar field $\tau(x)$. If we apply this to $R\bar{R} = 0$, we find directly

$$\partial\tau(x) = \frac{R}{\langle R\bar{u} \rangle_S} .$$

The evaluation of the field is now straightforward:

$$\begin{aligned} \mathbf{F}(x) &= \frac{K_0 e}{\langle R\bar{u} \rangle_S^3} \left(\langle R\bar{u} \rangle_V + \frac{1}{2} R \bar{u} u \bar{R} \right) \\ &= \mathbf{F}_c + \mathbf{F}_r \end{aligned}$$

and has a simple interpretation: \mathbf{F}_c is the boosted Coulomb field and \mathbf{F}_r is a directed plane wave propagating in the direction $\hat{\mathbf{R}}$ that is linear in the acceleration. Both \mathbf{F}_c and \mathbf{F}_r are simple fields, with \mathbf{F}_c predominantly electric, $\mathbf{F}_c \cdot \mathbf{F}_r = 0$, and \mathbf{F}_r a null flag. The spacetime plane of \mathbf{F}_c is the plane of $\langle R\bar{u} \rangle_V = \gamma(\mathbf{R} - \mathbf{v}R^0 - \langle \mathbf{R}\mathbf{v} \rangle_V)$. The real part gives the electric field in the direction of \mathbf{R} from the retarded position of the charge, minus

\mathbf{v} times the time R^0 for the radiation to get from the charge to the field point. The electric field thus points from the instantaneous “inertial image” of the charge, that is the position the charge would have at the instant the field is measured if its velocity remained the constant. If the charge moves at constant velocity, the electric field lines are straight from the instantaneous position of the charge. It is obvious that this had to be that way: in the rest frame of the charge the lines are straight from the charge, and the Lorentz transformation is linear and transforms straight lines into straight lines. The radiation field in the constant-velocity case vanishes, of course, because there is no acceleration. The magnetic field lines are normal to the spatial plane $\langle \mathbf{R}\mathbf{v} \rangle_V$ swept out by \mathbf{R} as the charge moves along \mathbf{v} . They thus circle around the charge path. With APS, we easily decipher both the spacetime geometry and the spatial version with the same formalism.

When a charge accelerates, its inertial image can move rapidly, even at superluminal speeds, and lines need to shift laterally. The Coulomb field lines are broken, and \mathbf{F}_c no longer satisfies Maxwell’s equations by itself. The radiation field \mathbf{F}_r is just the transverse field needed to connect the Coulomb lines. Only the *total* field $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_r$ is generally a solution to Maxwell’s equation.

The APS is sufficiently simple that calculations such as the field of a uniformly accelerated charge are easily made and provide simple analytical results that help unravel questions about the relation of such a radiating system to the nonradiating charge at rest in a uniform gravitational field.[6] One can also simplify the derivation of the Lorentz-Dirac equation for the motion of a point charge with radiation reaction and lay bare its relation to related equations such as the Landau-Lifshitz equation.[22]

There are many other cases where relativistic symmetries can simplify electrodynamics, and in many of them, relativity at first seems to play no significant role. For example, the currents induced in a conductor by an incident wave is easily calculated, and the oblique incident case is simply related to the normal incident one by a boost. Similarly, wave guide modes can be generated by boosting standing waves down the guide. APS plays a crucial role in unifying relativity with vectors and simple geometry.

6 Quantum Theory

As seen above, classical relativistic physics in Clifford algebra has a spinorial formulation that gives new geometrical insights and computational power to many problems. The formulation is closely related to standard quantum formalism. The algebraic use of spinors and projectors, together with the bilinear relations of spinors to observed currents, gives quantum-mechanical form to many classical results, and the clear geometric content of the algebra makes it an illuminating probe of the quantum/classical in-

terface. This section summarizes how APS has provided insight into spin-1/2 systems and their measurement.

Consider an elementary particle with an extended distribution. Its current density $j(x)$ is related by an eigenspinor field $\Lambda(x)$ to the reference-frame density ρ_{ref} :

$$j(x) = \Lambda(x) \rho_{\text{ref}}(x) \Lambda^\dagger(x) . \quad (6.1)$$

This form allows the velocity (and orientation) to be different at different spacetime positions x . The current density (6.1) can be written in terms of the density-normalized eigenspinor Ψ as

$$j = \Psi \Psi^\dagger, \quad \Psi = \rho_{\text{ref}}^{1/2} \Lambda ,$$

and it is independent of *gauge rotations* $\Psi \rightarrow \Psi R$ of the reference frame. The momentum of the particle $p = mu = m\Lambda\Lambda^\dagger$ can be multiplied from the right by $\bar{\Lambda}^\dagger = (\Lambda^\dagger)^{-1}$ to obtain

$$p\bar{\Lambda}^\dagger = m\Lambda .$$

This is the *classical Dirac equation*. [20] It is a real linear equation: real linear combinations of solutions are also solutions. In particular, since $\rho_{\text{ref}}^{1/2}$ is a scalar, it also holds for $\Psi = \rho_{\text{ref}}^{1/2} \Lambda$:

$$p\bar{\Psi}^\dagger = m\Psi . \quad (6.2)$$

The classical Dirac equation (6.2) is invariant under gauge rotations, and its real linear form suggests the possibility of wave interference.

Consider the eigenspinor Ψ_p for a free particle of well-defined constant momentum $p = mu$. In this case, we can define a proper time τ for the particle, and we look for an eigenspinor Ψ_p that depends only on τ . The continuity equation for j implies a constant ρ_{ref} :

$$\langle \partial \bar{j} \rangle_S = 0 = \langle (\partial \rho_{\text{ref}}) \bar{u} \rangle_S = \frac{d\rho_{\text{ref}}}{d\tau} ,$$

Gauge rotations $\Psi \rightarrow \Psi R$ include the possibility of a time-dependent rotation R . For the free particle of constant momentum, we assume a fixed rotation rate, and using gauge freedom to orient the reference frame, we take the rotation axis to be \mathbf{e}_3 in the reference frame. Including this rotation, the free eigenspinor Ψ_p has the form

$$\Psi_p(\tau) = \Psi_p(0) e^{-i\omega_0\tau\mathbf{e}_3} .$$

Now the proper time is given explicitly in Lorentz-invariant form by

$$\tau = \langle u\bar{x} \rangle_S = \left\langle \frac{p}{m} \bar{x} \right\rangle_S ,$$

and therefore the free eigenspinor Ψ_p has the spacetime dependence

$$\Psi_p(x) = \Psi_p(0) e^{-i(\omega_0/m)\langle p\bar{x}\rangle_S \mathbf{e}_3}. \quad (6.3)$$

The gauge rotation is associated with the *classical spin* of the particle, and the spacetime dependence of Ψ_p (6.3) gives the behavior of de Broglie waves provided we identify $\omega_0/m = \hbar$.

A further local gauge rotation by $\phi(x)$ about \mathbf{e}_3 in the reference frame can be accommodated without changing the physical momentum p in (6.3) by adding a gauge potential $A(x)$ that undergoes a compensating gauge transformation (to be determined below)

$$\Psi_p(x) = \Psi_p(0) e^{-i\omega_0\tau\mathbf{e}_3} = \Psi_p(0) e^{-i\langle(p+eA)\bar{x}\rangle_S \mathbf{e}_3/\hbar}.$$

The real linear form of the classical Dirac equation (6.2) suggests that real linear combinations

$$\Psi(x) = \int a(\mathbf{p}) e^{-i\langle(p+eA)\bar{x}\rangle_S \mathbf{e}_3/\hbar} d^3\mathbf{p},$$

where $a(\mathbf{p})$ is a scalar amplitude, may form more general solutions for particles of a given mass m . Such linear combinations are indeed solutions to (6.2) if p is replaced by the momentum operator defined by

$$p\Psi = i\hbar\partial\Psi\mathbf{e}_3 - eA\Psi.$$

With this replacement, the classical Dirac equation (6.2) becomes Dirac's quantum equation. The gauge transformation in A that compensates for the local gauge transformation $\Psi \rightarrow \Psi e^{-i\phi(x)\mathbf{e}_3}$ is now seen to be

$$A \rightarrow A + \frac{\hbar}{e}\partial\phi(x). \quad (6.4)$$

The gauge potential A is identified as the electromagnetic paravector potential, and the coupling constant e is the electric charge of the particle. The gauge transformation (6.4) is easily seen leave the electromagnetic field $\mathbf{F} = \langle\partial\bar{A}\rangle_V$ invariant. The traditional matrix form of the Dirac equation follows by splitting (6.2) into even and odd parts and projecting both onto the minimal left ideal $\mathcal{C}\ell_3\mathbf{P}_{\mathbf{e}_3}$, where $\mathbf{P}_{\mathbf{e}_3} = \frac{1}{2}(1 + \mathbf{e}_3)$.

The Dirac theory is the basis for current understanding about the relativistic quantum theory of spin-1/2 systems. The brief discussion above suggests that its foundations lie largely in Clifford's geometric algebra of classical systems. This suggestion is explored more fully elsewhere,[11] where it is shown that the basic two-valued property of spin-1/2 systems is indeed associated with a simple property of rotations in physical space. Extensions to higher dimensional spaces have succeeded in explaining the gauge symmetries of the standard model of elementary particles in terms of rotations in $\mathcal{C}\ell_7$, [24] and extensions to multiparticle systems promise to demystify entanglement.

7 Conclusions

In the space of this lecture, we have only scratched the surface of the many applications of Clifford's geometric algebra to physics. There has been no attempt at a thorough review of the work, the extent of which may be gleaned from texts[4, 6, 9, 28, 29], proceedings of recent conferences and workshops[5, 25, 26, 27, 30, 31] and from several websites.[32] I have instead tried to illustrate the conceptual and computational power the algebra brings to physics. Many of its tools arise from the spinorial formulation inherent in the algebra and are familiar from quantum mechanics. Had the electrodynamics and relativity of Maxwell and Einstein been originally formulated in APS, the transition to quantum theory would have been less of a "quantum leap."

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