GEOGEBRA-NA 2011
Proceedings of the Second North American GeoGebra Conference:
Where Mathematics, Education and Technology Meet

Editors:
Dragana Martinovic, University of Windsor
Zekeriya Karadag, Tufts University
Doug McDougall, University of Toronto

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History of GeoGebra-NA

One of the results stemming from the First International GeoGebra conference held on July 14-15, 2009, in Linz, Austria was in organizational activities around planning regional conferences in Spain, Turkey, Argentina, South America, and Norway.

In an attempt to engage the communities of mathematicians, mathematics educators, and software developers in discussions around the potential of technology for learning and teaching of mathematics, the First North-American GeoGebra conference (GeoGebra-NA2010), held on July 28-29, 2010 in Ithaca, NY, laid the foundations for a series of conferences in North America and consequently to this conference in Canada. The idea is to hold annual conferences interchangeably in the US, Canada and Mexico.

Goals of GeoGebra-NA

The GeoGebra-NA 2011 focuses on applications of computer technology in mathematics education from K-16+. In particular, this conference provides a forum for exploring research, development and application processes in relation to:

- Teaching Mathematics with Technology
  - Dynamic mathematics in action-educational research and experience
  - Applying multiple representations with GeoGebra
  - Promoting conceptual understanding through explorations.

- Learning Mathematics with Technology
  - Maximizing effectiveness of the software interface
  - Classic, home schooling and informal education
  - Bridging digital divide in access and learning opportunities.

- Implementation of Advanced Technologies in Mathematics Education
  - Developing big mathematics ideas through technology use
  - Using collaborative spaces to teach and learn mathematics
  - Making intra and interdisciplinary connections through technology use.
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Welcome from the Chairs:

Welcome to the Second North-American GeoGebra conference and the first GeoGebra conference held in Canada. We are excited with the prospect of this conference to build, support and sustain a community of mathematicians, mathematics educators, classroom teachers, graduate students and software developers who can benefit from the potential of computer applications in mathematics education.

The GeoGebra-NA 2011 organizers received over 30 papers from 10 countries and 5 proposals for the working groups. From this list, the reviewers selected 10 papers that found place in these proceedings.

We wish you all a pleasant stay in Toronto, many opportunities for intellectual exchange with the colleagues and much success in your professions.

We are also pleased to announce that GeoGebra-NA 2012 will be held in Mexico and hope that we will all meet there again!

Dragana, Doug, and Zekeriya
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Dr. Viktor Freiman is an Associate Professor at the Faculty of Educational Sciences, Université de Moncton. He teaches undergraduate courses in Mathematics Education and graduate courses in learning and technology, innovative trends in education and education of gifted. His doctorate study was about teaching methods in computer science courses. His recent research interests are related to problem-based interdisciplinary learning, mathematical giftedness and enrichment using technology. He is co-founder of the learning multidisciplinary community CAMI (www.umoncton.ca/cami) that offers problems and questions in mathematics, science, social studies, reading and chess. He is the author of articles, chapters and books and co-organiser of conferences and working groups on these issues in Canada and abroad. He is the president of the Association of the pedagogical advancement of technology in the Atlantic Canada APTICA (Avancement Pédagogique de TIC en Atlantique, www.aptica.ca).

Dr. Daniel Jarvis is an Associate Professor and Chair of the Master of Education Program at the Schulich School of Education, Nipissing University, North Bay, Ontario. With a background in mathematics and visual arts, Dan is particularly intrigued with the coalescence of these two disciplines via modern technology. His research interests include curricular integration, instructional technology, and educational leadership. Dan was invited to attend the first meeting of the International GeoGebra Institute in Cambridge, UK in 2008, and has been involved with IGI and the establishment of the GeoGebra Institute of Canada (GIC-IGC).
Walking through the universe of the World Wide Web, one can be amazed with the multiple-digit number of educational resources that may enhance teaching and learning to which anyone can get an access anytime and anywhere, once equipped with an appropriate technology, like for example, an individual laptop (Lowther, Ross, & Morrison, 2003). Namely, a Google-enhanced search for ‘new learning with GeoGebra returns more than 1 000 000 records. One of them would bring you to a project with the striking simplicity and infinite richness in terms of learning and teaching opportunities (http://livebinders.com/play/play/27560):

This screenshot unfolds many features of new techno-pedagogy that is based on ideas of knowledge building and knowledge sharing (Scardamalia & Bereiter, 2006), online collaborative (Roberts, 2004) and problem-based (Savery & Duffy, 2001) learning, as well as connectivism and connected knowledge (Downes, 2008).

Entering this space via the LiveBinders (beta) Knowledge Sharing Place (http://livebinders.com/shelf/search_display_author?terms=Knowledge+Sharing+Place), we can find this authoring binder (the nickname of the author and the date of creation are shown giving access to the author’s profile) that present a project assignment allowing students ‘learn the
GeoGebra program and create resources for others to learn GeoGebra (students teach other students), as shown on the above Figure. The menu on the top of the screen contains, among others, the entry to the GeoGebra community website, a YouTube sample of the activity, and news about new software’s developments. There is also a number of tools allowing members of this virtual learning community (Palloff & Pratt, 2005) to create their own profile, to log in, to rate the webpage, to share it, to twit, or to put it on the Facebook, as well as other related networks.

By focusing on mathematics knowledge building and knowledge sharing while working collaboratively as learning community, which is one of the striking characteristics of the Net Generation’s (Oblinger & Oblinger, 2005) innovative 21st century pedagogy (P21, 2011).

Based on innovative frameworks that define learning for tomorrow’s citizens, we shall reflect on following questions:

- What is this new way of learning?
- How to grasp and increase its potential for more differentiated and enriched learning?
- How can we deal with variety, complexity and messiness of new knowledge constructed in collaborative ways?
- What are indicators of success in such environments and how it can affect teaching and assessment practices?

These questions will outline our reflective thoughts about lessons GeoGebra learning community teaches us and how these lessons may enrich mathematics which is taught and learned.

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GEOGEBRA AND THE GLOBAL VILLAGE-THEATRE:
REALIZING THE 21ST CENTURY POTENTIAL OF DYNAMIC MATHEMATICS FOR/BY ALL

Dr. Daniel Jarvis
Schulich School of Education, Nipissing University
danj@nipissingu.ca

Abstract

In his visionary masterpiece, Understanding Media: The Extensions of Man (1994/1964), Canadian educator, philosopher and scholar, Marshall McLuhan, introduced his profound concept, “the medium is the message,” and also elaborated on how the world had been contracted into a “global village” by electronic technology. He would later extend this idea to “global theatre,” in which he emphasized the changeover from “hot media” (instructional) to “cool media” (participatory), from consumer to producer, from acquisition to involvement—in the contemporary vernacular, from Internet to Web 2.0. This presentation will highlight the history, growth, and potential of GeoGebra (2001-2011) as a mathematics education software that has clearly showcased the 21st century realities of global village and theatre, providing free access and open participation, respectively, to millions of users worldwide.

The Medium is the Message

The year 2011 marks the 100th anniversary of the birth of Dr. Marshall McLuhan (1911-1980), Canadian educator, philosopher, scholar, and, some would argue, post-modern prophet. In his 1964 book, Understanding Media: The Extensions of Man, McLuhan explained how that with the advent of each new major technology not only does the way in which information is communicated and received change, but there are also significant changes in the way individuals and society react to the new technology itself.

In a culture like ours, long accustomed to splitting and dividing all things as a means of control, it is sometimes a bit of a shock to be reminded that, in operational and practical fact, the medium is the message. This is merely to say that the personal and social consequences of any medium—that is, of any extension of ourselves—result from the new scale that is introduced into our affairs by each extension of ourselves, or by any new technology. (McLuhan, 1994/1964, p. 7)

Educational software, like GeoGebra, affords students new ways of thinking about, manipulating, and analyzing mathematical models. Highspeed Internet and social networking software further
provide new and easily available/affordable ways of sharing these learning episodes with others in community.

**The Global Village**

McLuhan also described the dramatic reversal, or social “implosion” of the 1960s era as one in which pervasive electronic media had, in essence, shrunk the known world, thereby increasing the connectivity of individuals, and hence their perceived social accountability towards others. He boldly proclaimed (p. 5),

> In the electric age, when our central nervous system is technologically extended to involve us in the whole of mankind and to incorporate the whole of mankind in us, we necessarily participate, in depth, in the consequences of our every action. . . . After three thousand years of specialist explosion and of increasing specialism and alienation in the technological extensions of our bodies, our world has become compressional by dramatic reversal. As electrically contracted, the globe is no more than a village. (p. 5)

Although the GeoGebra user community was growing astonishingly fast, the core team realised in 2007 that teachers who were beginning to implement the software, as well as researchers who were starting to document its usage, needed specific forms of support and direction. In addition, they wanted to offer a forum for the community to extend collaboration and communication. To be able to offer such assistance, they established the International GeoGebra Institute (IGI). The aims of this non-profit organisation were to assist all members of the international GeoGebra community based on their own local context and needs. During the past several years, IGI and the GeoGebra community have been gaining substantial momentum, and GeoGebra is rapidly gaining popularity in the teaching and learning of mathematics around the world. Currently, the software is translated into 52 languages, used in 190 countries/regions, and downloaded by approximately 300,000 users each month. McLuhan’s global village has been evidenced in this context.

**On Multiple Representations**

Modelling mathematical phenomena with different tools represents a powerful way for students to visualize, and hence better understand the big ideas of mathematics (Hohenwarter & Jones, 2007). When various digital models converge and are actually connected by means of an underlying code, we can imagine that the effects are even more remarkable for many students (and teachers). McLuhan (1994/1964, p. 13) highlighted Picasso’s Cubism (which shattered an object into broken pieces of multiple perspectives, and then reassembled them by way of visual
collage to provide for the viewing experience what Pablo argued was a more “real” way of seeing) as a powerful metaphor for detailed life observations.

The message of the movie medium is that of transition from lineal connections to configurations. . . . Instead of the specialized illusion of the third dimension on canvas, cubism sets up interplay of planes and contradiction or dramatic conflict of patterns, lights, textures that “drives home the message” by involvement. This is held by many to be an exercise in painting, not in illusion. In other words, cubism, by giving the inside and outside, the top, bottom, back, and front and the rest, in two dimensions, drops the illusion of perspective in favor of instant sensory awareness of the whole. Cubism, by seizing on instant total awareness, suddenly announced that the medium is the message.

GeoGebra offers “next step” interactive, multi-representational software (Todd, Lyublinskaya, & Ryzhik, 2010), which capitalizes on the complexity of the electronic age, and on the mathematical minds at work.

On Innovation & Collaboration

Some have argued that an open-source, freely-accessible software could not sustain itself at length without the type of guaranteed remuneration/support offered to employees of large software corporations (for a related discussion, see Jarvis, Hohenwarter, & Lavicza, 2011). McLuhan explained how, ironically perhaps, innovation actually “threatens the equilibrium of existing organization,” how internal mechanisms are actually created to deal with this perceived internal threat, and how innovations often come from outside:

In big industry new ideas are invited to rear their heads so that they can be clobbered at once. The idea department of a big firm is a sort of lab for isolating dangerous viruses. When one is found, it is assigned to a group for neutralizing and immunizing treatment. . . . Therefore, no new idea ever starts from within a big operation. It must assail the organization from outside, through some small but competing organization. (p. 251)

During the past several decades, it has been demonstrated that a large number of enthusiasts can alter conventional thinking and models of development and innovation. The success of open-source projects like Linux, Firefox, Moodle, and Wikipedia has shown that collaboration and sharing can produce, perhaps quite surprisingly at first, valuable resources in a variety of areas of life. While working on the open-source project GeoGebra, Hohenwarter and his core team have witnessed the emergence of an enthusiastic, international community. New advances such as the development of GeoGebra CAS (Computer Algebra Systems), GeoGebra Spreadsheet, GeoGebra Mobile (for smartphones), GeoGebra 3-D, GeoGebra XO (one-laptop-per-child
initiative), and GeoGebraTube (for better tagging/searching of shared resources), and the release of GeoGebra 4 (August, 2011), are in large part attributable to the contributions of various volunteer individuals or teams found within this growing nexus of engaged participation.

McLuhan envisioned a world of electronic connectivity, instantaneous information sharing, freely-available resources, self-tailored education, and a finely-tuned (although not necessarily harmonious) collective consciousness. The medium of the Internet has subsequently become a powerful message with Google, Wikipedia, Facebook, Twitter, email, smartphones, and touch tablets literally changing the way we work, play, spend, and socialize. GeoGebra, as a mathematics education software phenomenon, will undoubtedly continue to constitute some significant part of our shared global village-theatre in the 21st century world.

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Introduction

In this paper we explore the use of GeoGebra as a visual dynamic tool for supporting high school students’ understanding of probability, more specifically, understanding of conditional probability and Bayes’ theorem. The research presented is part of a broader research project on improving high school students’ risk literacy. Decisions that involve the understanding of risk are made in all aspects of life including health (e.g. whether to continue with the course of medication), finances (paying for extra insurance) and politics (preemptive strikes versus political dialogue). These decisions are not only common, but they are also critical for individual and societal health and well being. Some studies have shown that people are routinely exposed to medical risk information (e.g. prevalence rates of diseases) and that their understanding of this information can have serious implications on their health (Rothman et al., 2008). Despite its importance, most people are unable to adequately interpret and communicate risk (Reyna et al., 2009).

Theoretical Framework

Risk literacy

Although there is a recognized urgent need for risk literacy education, there is a lack of agreement on its definition. This is because the concept or risk literacy exists within the intersection of many related literacies—mathematical, health, statistical, probability, scientific, and financial. In this study, we will situate risk literacy within the fields of statistical and probability literacy as both fields focus on uncertainty and chance which are important elements of risk literacy. Most current approaches to literacy recognize it to be more than a minimal subset of content knowledge in a particular field. Further, the definition of literacy has been expanded to include “desired beliefs, habits of mind, or attitudes, as well as a general awareness and a critical perspective” (Gal, 2004, p. 48). Consistent with Gal’s (2004, 2005) research on statistical and probability literacy, we will define risk literacy to consist of knowledge (literacy skills, understanding of math, probability and statistics, specific content knowledge, etc) and dispositional elements (beliefs and attitudes and critical stance). In this paper, we are concerned with knowledge elements, more specifically, conditional probability and Bayes’ theorem.
Understanding conditional probability

Kahneman, Slovic & Tversky (1982) claimed that intuitive errors proceed from using certain heuristic principles that often lead to erroneous probability judgments. For instance, according to the principle of representativeness, people assess the probability of an event according to the extent to which this event’s description reflects the way they perceive the set of its most likely consequences, or alternatively, the process that produces the event. As the consequence of the representativeness heuristic, people tend to neglect the base rates which are, according to Bayes' theorem, relevant to the calculation of probabilities. In addition, Koehler (1996) describe and provide empirical evidence for inverse fallacy in which the conditional probability of the event A given the event B is taken to be equivalent to the conditional probability of the event B given the event A.

Dynamic visualizations

An enormous corpus of literature has been accumulated on technology use in mathematics education and visual learning of mathematics, and many researchers discussed the topics in the various national and international meetings such as PME, PMENA, and ICMI (Arcavi, 1999; Duval, 1999; Hitt, 1999; Hoyles, 2008; Kaput & Hegedus, 2000; McDougall, 1999; Moreno-Armella, 1999; Presmeg, 1999; Santos-Trigo, 1999; Thompson, 1999). In the last couple of years there has been a focus on dynamic learning environments such as GeoGebra which allow us to create mathematical objects and explore them visually and dynamically.

Moreno-Armella, Hegedus, and Kaput (2008) describe learning environment in which students can visualize, construct and manipulate mathematical concepts. The dynamic learning environments can enable students to act mathematically, to seek relationship between object that would not be as intuitive with a static paper and pen representations.

Research Objective and Questions

The goal of our research is to substantiate the claim that dynamic learning environment enables student to grasp abstract mathematical concept by manipulating mathematical objects constructed within these systems. More specifically, we investigate the role of dynamic visualization of Bayes' theorem in students' understanding of the theorem. Moreover, we are interested whether the introduction of visualization has an effect on student’s committing the base rate and the inverse fallacy.

Methods

The design experiment is an iterative process consisting of assessment and instructional intervention (Cobb et al., 2003). Through iterative steps, the assessment and the intervention inform
each other. The goal of the process is to improve the instruction as well as to gain insight into students' learning processes (Cobb et al., 2003). The design research took place in a grade 11 classroom during the probability and statistics unit. There were 23 participants. As a part of the initial assessment, students were presented with the breast cancer problem. Only two students out of 20 who participated in the initial assessment were able to give the correct answer to the question. After the initial assessment students, were presented with the GeoGebra applet representing area-proportionate Venn diagrams whose size can be manipulated using the slider feature of Geogebra. Students’ interaction with the software was audio recorded together with the class discussion facilitated by the researchers. Finally, the post-intervention test presenting a problem analogous to the breast cancer problem but in a different context was presented.

![GeoGebra applet for visualizing Bayes' theorem](image)

**Visualizing Bayes Theorem**  
Nenad Radakovic, OISE/University of Toronto

**Results and Conclusion**

Evidence from the interviews suggests that area-proportional Venn diagrams enabled students to conceptually understand the Bayes' formula. Using the dynamic feature of the applet, the students were able to understand the connection between the base rates and conditional probability. The results of the post-test shows that 75% of students were able to solve the question equivalent to the breast cancer problem. This serves as evidence that dynamic visualizations do create means for the deeper analysis of mathematical concepts. Finally, the study illustrates a role...
that a dynamic learning environment such as GeoGebra could play in fostering students’ risk literacy.

References


SIGNIFICANT TEACHING AND LEARNING MOMENTS USING INTERACTIVE WHITEBOARDS IN MATHEMATICS

Dr. Catherine D. Bruce, Rich McPherson & Farhad M. Sabeti
cathybruce@trentu.ca

Conference Theme: Computer technologies for learning/doing mathematics

Key Words: Mathematics, Interactive Whiteboards, Teacher and Student Learning, Gestures

Context

How are interactive whiteboards (IWB) helping students learn mathematics? Students report being engaged, but are they gaining deeper understanding of concepts explored? Our research goal was to define and analyze significant teaching and learning moments using the IWB (moments that had a clearly positive impact on student understanding because of IWB use) in two classrooms. 25 classroom video episodes were systematically analyzed using an analysis matrix. We conclude that deep student understanding was achieved where: 1. Students illustrated their thinking using tools available on the IWB; 2. Students took risks in pairs activities, investigated multiple solutions, and efficiently completed tasks using IWB technology; 3. Student use of gestures enabled consolidation of ideas.

Results are based on two case studies conducted with two Grade 6 classrooms and their teachers. Both teachers were chosen based on their distinct levels of experience and knowledge of interactive whiteboard technology: one teacher having 12 years of teaching experience with no interactive whiteboard experience, and one teacher with 3 years of teaching experience and relatively high knowledge of interactive whiteboard use in mathematics.

Literature Review

Benefits of IWB use identified through early studies included: a) ease of use for whole class teaching (Stephens, 2000) including dynamic visual demonstrations (Kennewell & Beauchamp, 2003); b) classroom management through IWB-generated engagement; and c) the integrated use of a range of multimedia resources (Ekhami, 2002). After almost a decade of research, student engagement has received the greatest research interest as students interact with a technological tool that is considered to be relevant to their understanding.

Interestingly, the use of gestures is a critical component of IWB use. Singer and Goldin-Meadow (2005) reported that gestures carry a unique learning potential; they found that students were
able benefit from teachers’ use of gesture when the strategy conveyed through gesture differed from the strategy conveyed through speech. Broaders, Cook, Mitchell, and Goldin-Meadow (2007) found that students encouraged to use gestures were more likely to be cognitively receptive to future mathematics learning than those not encouraged to use gestures. Given that much of the input on an IWB directly employs one’s hands and fingers, the potentially unique role of gestures within the interactive environment of IWB technologies is worth examining.

A key concern noted in recent research by Slay, Sieborger and Williams (2008) is that teachers are not given sufficient supports to integrate this new technology with inquiry-based mathematics pedagogy. During the familiarization phase, the IWB is used as a static device and little evolution of teaching practice is taking place. Inevitably, while teachers learn to use the IWB as an instructional tool, the lesson focus returns to the teacher (Holmes, 2009) including interruptions to the flow of lessons and/or student thinking while teachers and students troubleshoot.

Nonetheless, there is a common thread in the research literature indicating that IWB use has the potential to reinforce lesson participation and peer interactivity within classrooms, but more importantly, that effective IWB use begins with the teacher and their Technological Pedagogical Content Knowledge (TPCK) (Holmes, 2009). Our goal in this study was to explore the research question: What does effective use of the IWB look like in the mathematics classroom?

**Methods**

Video data and field notes in two classrooms, as well as interviews with students and teachers were collected over nine months. After initial reviewing of all data sets, the data were grouped into 3 broad categories: Positive learning episodes with the IWB (239 episodes), negative learning episodes with the IWB (78 episodes) and neutral moments (8 episodes). Researchers then developed criteria for analyzing video, of what constituted a significant teaching and learning moment with the IWB. The criteria for selection were: clarity of visual and audio of episodes and instances where IWB use was most active.

Episodes that met the above criteria were then coded. All codes were counted and subsequently clustered into two natural groups: (i) Illustrations of thinking using IWB tools and (ii) justifications and communication of mathematics ideas using the IWB (see Table 1).
Table 1. Significant teaching and learning moments from the 3 primary data sources

<table>
<thead>
<tr>
<th>CODE COUNTS of Significant Teaching and Learning Moments</th>
<th>Field Notes</th>
<th>Interview</th>
<th>Video</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Illustration of Thinking Using Tools</strong></td>
<td>208</td>
<td>272</td>
<td>4:11:47</td>
</tr>
<tr>
<td>Use of Tools (UT)</td>
<td>107</td>
<td>165</td>
<td>1:47:59</td>
</tr>
<tr>
<td>Justifications of Solutions using Tools (JST)</td>
<td>41</td>
<td>57</td>
<td>1:18:49</td>
</tr>
<tr>
<td>Consolidation of Ideas using Tools (CIT)</td>
<td>60</td>
<td>50</td>
<td>1:04:59</td>
</tr>
<tr>
<td><strong>Benefits of Pairs Work (&amp; Whole Grp)</strong></td>
<td>128</td>
<td>167</td>
<td>2:01:03</td>
</tr>
<tr>
<td>Multiple Solutions at IWB(MSP)</td>
<td>41</td>
<td>62</td>
<td>54:19</td>
</tr>
<tr>
<td>Risk-taking at IWB (RTP)</td>
<td>29</td>
<td>26</td>
<td>27:10</td>
</tr>
<tr>
<td>Efficiency of IWB (EP)</td>
<td>58</td>
<td>79</td>
<td>39:34</td>
</tr>
<tr>
<td><strong>Total Counts</strong></td>
<td>336</td>
<td>439</td>
<td>6:12:50</td>
</tr>
</tbody>
</table>

A third cluster of analysis was exclusive to video data: the use of gestures with the IWB that mediated understanding. For this analysis, four short episodes from the full video data set were selected to explore and illustrate the role of gestures within IWB-mediated mathematics learning.

**Findings and Discussion**

The most robust finding in our study was that the Interactive Whiteboard supported teacher and student communication of their mathematics thinking when they used the tools available. Students working at the IWB would often use the statements such as “let me show you”. The use of IWB tools, such as dragging an angle measuring tool from the IWB gallery onto the screen for large screen use, enabled efficient and visually explicit explanations. In particular these tools enabled student justifications of ideas, and consolidation of mathematics concepts. When given the opportunity to use the tools, students were observed being consistently engaged, asking deep/productive questions that challenged their peers, and generating rich discussion that moved the learning forward.

Analysis also showed that the most effective instances of IWB use occurred when students were encouraged to work together in pairs and in some whole group scenarios. Pairs of students working at the IWB were observed as focused, taking risks, and often challenging their partners’ thinking when looking for solutions; in contrast with students at their desks in these same lessons who took less risks and engaged in less mathematical debate during discussions. Whole group learning at the IWB was similarly powerful in that it provided the teacher with an opportunity to use the dual-screen tool to showcase two student IWB work samples simultaneously for discussion and
debate. The data illustrate that during these episodes, students became responsible for generating discussion in a community of learners. Student interview data confirmed that the ability to see a variety of IWB generated solutions at once was powerful for deepening student understanding. “It showed their work and everyone else could come up and share their work and you could compare the two and it could help you learn multiple ways of seeing something” (Student, June 20, 39-41).

Gestures were used by students and their teachers in a variety of contexts to communicate a broad scope of information. All but one of the gestures observed were employed by both the teacher and the students and there were several instances where the use of gestures by one member of the classroom prompted further use of gestures by others. For example, in representing 360°, the teacher used her index finger to trace out a complete circle in front of her students. In this and other cases, the teacher and her students also employed the IWB to complement their use of gesture.

While the design of this study was not intended to reveal causal relationships, the data suggests the ability of gesture to support and facilitate learning in the presence of an IWB. Throughout the episodes, students employed gestures to communicate and further their mathematical reasoning in several ways. In the first episode, gesture was used to communicate the properties of solids by representing faces and their relationships. In the second episode, tools on the IWB and their associated gestures for rotation helped a pair of students classify solids according to their bases. Also in this episode, a student used a gesture to communicate the shape of a figure on the IWB prior to describing its shape in words. This phenomenon recurred in the third episode when a student used a gesture, in the absence of words, to communicate a line graph. In the final episode, the teacher employed gestures as a vehicle to relate multiple representations of an angle drawn on the IWB. These examples illustrate that gesture can be an actively used and an effective tool to support an IWB-mediated learning environment.

Nonetheless, a number of limitations emerged. For example, students became frustrated with the layering of objects within the IWB software. The students in one episode particularly struggled with this issue when they moved shapes into their constructed Venn diagram, only to find that object no longer ‘selectable’ due to the order of objects in layers. Although this limitation resurfaced repeatedly, most students were able to circumvent this by using the ‘lock’ feature to secure the position and layering of one or more objects on the screen. A number of other limitations and negative examples will be further discussed in the full paper and at the conference. The research presentation will feature videos clips of observed lessons, student and teacher interviews, along with summaries of results. Participants will be asked to evaluate one or two significant teaching
and learning moments with analysis tools in order to generate discussion about the strengths and perils of IWB use in mathematics.

References


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MATHEMATICAL PROBLEM SOLVING IN DYNAMIC LEARNING ENVIRONMENTS:  
THE CASE OF GEOGEBRA

Linggou Bu¹, Erhan Selcuk Haciomeroglu², Frackson Mumba³

¹,³Southern Illinois University, Carbondale, Illinois, USA.
²University of Central Florida, Florida, USA

¹lgbu@siu.edu, ²erhansh@mail.ucf.edu, ³frackson@siu.edu

Abstract

Emerging dynamic mathematics software supports mathematics learning environments that are rich in multiple representations, modeling and simulation tools, computational utilities, and pedagogical innovations. These resources provide new opportunities for and challenges to mathematical problem solving. Grounded in existing literature on mathematical problem solving, this article presents heuristic strategies that are specifically applicable when students engage in meaningful mathematical problem solving using the open-source software GeoGebra or similar technologies. These strategies, illustrated with specific problems, include exploring and planning, understanding dependency, working backwards, guessing and checking, and managing complexity.

Introduction

Mathematical problem solving is a complex process. In solving a problem, students need to coordinate a variety of resources, heuristics, and self-regulation skills (Polya, 1945/2004; Schoenfeld, 1985). Meanwhile, their problem solving efforts are constrained and supported by the specific context, their belief system, and their mathematical world view (Schoenfeld, 1985, 2008). Emerging interactive mathematics learning technologies provide dynamic multiple representations, computational utilities, and modeling mechanisms that can be pedagogically beneficial in facilitating students’ problem solving efforts. Among such technologies is the open-source learning environment GeoGebra (www.geogebra.org), which integrates algebra, geometry, and calculus packages in a connected system of mathematical content and pedagogical tools (Hohenwarter & Hohenwarter, 2009). In such technology-based dynamic environments, traditional problem solving strategies are generally informative but are not adequate in supporting student-initiated problem exploration and construction. The dynamic nature of problem representation and learner control provides both new opportunities for and new challenges to meaningful problem solving and the effective teaching of problem solving. For instance, in a dynamic environment, students can explore ready-made mathematical models to identify invariant relations in a problem situation; furthermore, they can create their own
mathematical models to address a problem situation or modify and extend existing ones as a way to learn about the underlying mathematical structure.

Informed by existing literature on problem solving and, especially, a model-centered perspective on problem solving (Lesh & Doerr, 2003), we discuss through examples a few heuristics, or rules of thumb, that are of primary importance for solving problems using GeoGebra—exploring and planning, understanding dependency, working backwards, guessing and checking, and managing complexity. These strategies may be generally helpful in the case of similar mathematics learning technologies.

**Heuristic Strategies**

**Exploring and Planning**

In a dynamic mathematics learning environment, it is possible and pedagogically necessary to have students explore the mathematical ideas. Such explorations can help students understand a problem and/or its components, get acquainted with the resources at hand, and further establish a plan of action. Such a plan will not only help students construct a tentative solution, but also will serve as a point of reflection (Polya, 1945). Among the major affordances of dynamic mathematics are representational flexibility and interactivity, which provide potential insight into the problem situation with instant feedback to students through on-screen invariant patterns and graphic and/or algebraic updates. Frequently, students tend to expect, unrealistically, such a plan to arise from the dynamic technology environment, although such planning should be an essential part of their problem solving efforts and thus must be a result of their mathematical reasoning in relation to the problem. Therefore, as the first heuristic strategy, we recommend that students develop an explicit action plan, albeit inadequate initially, before they start to solve a given problem. There is much more involved in the process of planning a solution, which we discuss in the following sections.

For example, as students attempt to show that the area of a triangle can be uniquely determined by its base and its altitude, we could start by asking them to construct and play with an arbitrary triangle, identifying a base and the corresponding altitude. We found that many students were not aware of the meaning of arbitrary in a mathematical context, although it is an important aspect of dynamic mathematical constructions. This interactive exploration of triangles will help students recall facts, clarify related concepts, appreciate the underlying mathematical relations, and set the stage for higher levels of reasoning. In making a plan to solve the problem, we suggest that students first start with a fixed base; second, find a way to fix the altitude (such as placing the third vertex on a line parallel to the base); third, make a triangle and measure its area; fourth, change the specific shape of the triangle without changing the base or the altitude by shifting the
point C on L1 (see Figure 1); and finally, manipulate the base and/or the altitude for more cases. Initially, such a plan is just a hypothetical trajectory of construction, along which students may need to make adjustments when they come to know more about the problem itself and the tools. To make a feasible plan and avoid unnecessary difficulties, students will also need to understand the interrelationships among various elements of a dynamic construction, which we discuss in the next section.

Figure 1. Exploring the area of a triangle in relation to its base and altitude.

**Understanding Dependency**

In a dynamic mathematics learning environment, objects are interrelated through explicit links and/or the order of their construction. In GeoGebra and similar mathematics software, there are two types of mathematical objects: **free (or independent) objects** and **dependent objects**. Free objects are the starting points of subsequent mathematical relations, and thus can be freely dragged or manipulated, while dependent objects rely on one or more free objects or other dependent objects, and therefore cannot be freely manipulated. Such a relationship is referred to as **dependency** (Jones, 1996) and exists among both geometric and algebraic objects.

It is worth noting that in a traditional paper-and-pencil environment, there are also free objects and dependent objects, but the dependency remains implicit to the audience. For example, when we draw a line segment connecting two points A and B, the points are free objects and the segment will be dependent on the points A and B. The perpendicular bisector of the segment AB will also be dependent on A and B. In other words, if we change the positions of A and B, both the
segment AB and its perpendicular bisector will change accordingly. However, such relations are not so significant pedagogically in a traditional setting or on paper. By contrast, in a dynamic environment, the idea of dependency is crucially important for students to manipulate initial conditions of a problem and draw general conclusions. In general, any mathematical construction, including its multiple representations, is a web of dynamic relations, which, on the one hand, are the foundation for open-ended exploration, and on the other, new mathematical relations to be mastered by students. Students must come to understand the dependencies in a problem solving situation in order to devise an effective plan of action. To solve a problem, therefore, one must understand its dependencies, which, once again, are a critical part that calls for mathematical reasoning on the part of the problem solver.

Referring to the triangle area problem (Figure 1), there are three free objects: Points A, B, and D. Points A and B determines the base; point D sets the altitude. The line L1 goes through point D and is parallel to segment AB, and is thus dependent on points A, B, and D. Point C is dependent on line L1, and thus can move freely on L1. There are other solutions to the triangle problem. For example, one does not have to use a parallel line to constrain the altitude. Nonetheless, similar dependencies exist and are crucial to building a dynamic construction to model and then solve a problem.

As a second example, we look at the following problem: Mary looks at herself in a plane mirror and she wonders why her feet in the mirror seem to be at the same distance to the mirror as her real feet. A direct modeling method could be used to solve Mary’s problem, given the fact that if she sees her feet in the mirror, there has to be some light going from her feet to the mirror and then bouncing back to her eyes. As shown in Figure 2, the construction could start with points A and B, which define the mirror line L1. Point C represents the position of the eye. Realistically, the eye is rather a ball than a mathematical point. Thus, we can draw a circle of a certain radius to represent the eye. Since the person should stand in parallel to the mirror, we could draw a line L2 through point C that is parallel to line L1. L2 represents Mary and point D represents the position of her feet. Note that points A, B, C, and D are all free objects, which set up the mirror and the distance between the mirror and the person.
Figure 2. How light travels from the object to the eyes.

A light ray from the feet (point D) reaches the mirror and is reflected at some point on line L1. For that purpose, we pick a point F on line L1. Thus, point F is dependent and can be freely moved on the line L1. Subsequently we draw a segment DF for a light ray that goes from point D to the mirror at point F. Segment FD’ is the reflected light ray corresponding to DF. To the eye, the image of the feet (point D) should be on the extension of D’F. Therefore, we draw a line connecting D’ and F as shown by the dotted line (Figure 2).

With the dependencies resolved, we can solve Mary’s problem. If there was only one ray of light that leaves the feet and is bounced to the eye, then the eye could not uniquely locate the image of the feet. However, in reality, there are always multiple rays of light that are reflected to the eye. By dragging point F on line L1 and tracing the dotted line as shown in Figure 2, we could uniquely locate the image of the feet by a process called triangulation. A mathematical proof of the distances can be established by citing congruent triangles. There are, of course, other ways to identify and establish the dependencies, which will affect the subsequent explorations and conclusions.

Establishing dependencies in a GeoGebra construction is virtually synonymous to problem solving and calls for extensive problem analysis, including trials and errors. It is in the process of establishing dependencies that a student comes to understand the problem better and further develop a plan of action, which may further help him/her establish the dependencies. Such a process of problem solving is iterative in nature. The tools available in GeoGebra facilitate that process and offer an environment full of interactivity and prompt feedback.
Working Backwards

In working backwards, we assume that we know the solution to a given problem and then work back step by step to build the relationships between the solution and the given conditions, or, in other words, to establish the dependencies among the various components of a solution. It is one of the most powerful heuristics in problem solving, especially in geometric constructions (Polya, 1945/2004). One classic problem is to construct a parabola from its definition—a collection of points that are equidistant from a given point (the focus) and a given line (the directrix). The definition is easy to understand, but does not tell us how to construct such a curve. To develop a plan, we could assume that we have a solution already and work backwards. As shown in Figure 3, if point P is a point on the parabola, then it is equally distant to the directrix L1 and the focus F. To find such distances, we connect points P and F and draw a line L2 through P that is perpendicular to L1. Line L2 and line L1 intersect at point D. That, however, does not uniquely determine point P. Further analysis reveals that point P is on the perpendicular bisector of segment DF. Therefore, we make segment DF and its bisector.

The analysis above produces a series of dependent elements and leads to a plan of action for constructing a parabola from its definition. We now work backwards: Starting from the directrix L1, we pick a point D on it, and then draw a line L2 that is perpendicular to L1 at point D. We further connect points D and F and construct the perpendicular bisector L3 of segment DF. The intersection of line L2 and line L3 is the corresponding point P on the parabola. Since point D is movable on line L1, we could drag point D along L1 to trace the whole parabola.

Figure 3. Developing a plan for constructing a parabola by working backwards.
In working backwards, students could identify the intrinsic relationships or dependencies that constitute a parabola and further work out an action plan. In carrying out their plan, they can confirm, modify, or reject their emerging conceptions, where the dynamic environment provides instant feedback and tools for exploration and reflection.

**Guessing and Checking**

A mathematical problem is relative in nature. What is a problem to some students may be only an exercise or a direct recall of previous schemes of action to others. When solving a genuine problem, most students tend to work out an intuitive guess based on their prior knowledge and the surface structure of the problem (Willingham, 2009). Although such a guess may indeed be the correct answer, it is, more frequently, an indicator of their prior knowledge or even their misconception of a certain domain of mathematics (Ryan & Williams, 2007). In either case, such guesses, which are somehow plausible to problem solvers, must be addressed pedagogically in order to engage students in seeking a meaningful perspective on the problem. Polya contends that “if the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference” (1954, p. v). In a dynamic environment like GeoGebra, guessing and checking is an important strategy to explore the problem, clarify the conditions, understand the problem, and ultimately conceptualise some actions or processes that are appropriate for solving the problem.

As an example, consider the following scaling problem. If we start with a plane figure such as a triangle or rectangle and then double, triple, or quadruple the length of each side, how does its area change, respectively? Many students tend to have a persistent intuitive reaction to the problem, guessing that the area will double, triple, or quadruple accordingly. In a dynamic environment, such guesses can be easily checked using the dilatation tool, followed by other in-depth explorations. In Figure 4, for example, Triangle ABC is dilated by a linear factor (scaleFactor), which controls the amount of linear scaling. Point P is the ordered pair (linear factor, area factor). As can be seen, the area is quadratically related to the linear measures. Further, students could also take a guess at the effect of a negative scaling factor such as \(-2\) on the resulting figure and its area.
Figure 4. How does the area of plane figure change as it is dilated?

Because of the open-ended nature of dynamic environments, guessing and checking plays a pedagogically meaningful role in sustaining students' active mathematical thinking. As students take guesses at the problem solution and check the validity of such guesses, they will continue to learn about the problem itself and their own mathematical conceptions, identify the essential dependencies and develop a plan of action. Just as Polya (1981) points out, "[a]s the work of the problem solver progresses, the face of the problem continually changes" (p. 89). Furthermore, students could learn to pose new problems on the basis of their explorations to themselves or their peers in order to participate in internal conversations or social interactions.

Managing Complexity

Mathematical problem solving is a complex process, as discussed previously. On the one hand, the problem itself may be intrinsically complex, involving multiple components and their interrelations, which increases the cognitive load for students; on the other hand, the problem solver may have limited or incomplete knowledge about the problem or the resources in the dynamic environment and thus may face special cognitive challenges arising from the mathematical and technological aspects of the dynamic environment (Yoon, Thomas, & Dreyfus, 2009). In solving a problem using GeoGebra, it is important to manage both aspects of learning complexity and engage students in integrated and whole-task learning (Foshay, Silber, & Stelnicki, 2003; Milrad, Spector, & Davidsen, 2003).

To activate or enrich students' prior knowledge of the problem, we could allow them to use existing tools or provide them with ready-made dynamic worksheets for part-tasks or subproblems. For example, if the concept of slope is used in a comprehensive problem, students could use the
slope tool to find the slope of a function at a certain point. Similarly, if the Pythagorean Theorem is to be used in another problem, students could be provided with a ready-made worksheet to review the properties of a right triangle.

During the process of problem solving, the computer screen may get crowded with various intermediate objects. Such objects, if not essential to the subsequent steps, could be hidden behind the visible objects. For example, when we extend a segment in both directions, the original segment could be hidden to allow the learner to focus on the resulting line. If necessary, hidden objects can be made visible by referring to their algebraic representations in the system.

In helping students manage complexity, the teacher could provide half-completed worksheets as scaffolding for them to complete the whole task. Alternatively, user-defined button tools could be created to simplify or facilitate future operations. For example, if we want to trisect each side of a triangle for another instructional goal, we need to know how to trisect a segment and then repeat the same procedure three times, which could make the process unnecessarily tedious and visually confusing. In fact, we would trisect any segment, and then use the input and output of that procedure to define a TrisectSegment tool and then use the tool to trisect any segment (see Figure 5).

Finally, the complexity of a problem could be managed by using multiple representations in a dynamic way that allows automatic synchronization. For example, as students explore quadratic functions of the form, \( f(x) = ax^2 + bx + c \), we could present the functions in three different representations: an algebraic function, a geometric graph, or a tabular mapping. Further, these three representations could be dynamically linked; when one is changed, the other forms will be updated automatically without direct user input.
Figure 5. Managing complexity using a user-defined tool: the case of trisecting segments.

Conclusion

Mathematical problem solving is a challenging process, involving careful coordination of resources and heuristic strategies. In solving a genuine problem, students would be necessarily going through three stages: exploring and understanding the problem, verbalizing and formalizing the problem using multiple representations and mathematical models, and ultimately assimilating this experience into their whole mental outlook on mathematics (Polya, 1981). In a pedagogical sense, as students construct increasingly abstract models of the problem situations, they are also building their identity as persistent and skilful problem solvers (Goldin, 2007).

In this article, we discussed a few heuristic strategies that are helpful in solving problem in a dynamic environment like GeoGebra: exploring and planning, understanding dependency, working backwards, guessing and checking, and managing complexity. Although all of these strategies are applicable in traditional instruction, they take on more generative roles in a dynamic resource-rich environment, especially understanding dependency and managing complexity. Dynamic mathematics learning environments like GeoGebra afford a variety of technological tools and dynamic multiple representations that can potentially facilitate students’ problem exploration, computation, construction, modeling, and reflection. It is worth noting that the heuristic strategies we discussed here are general guidelines for solving problems using new mathematics learning technologies, the specific actions needed to solve a problem tend to come from an integrated perspective on both the mathematical content and technological tools (Doerr & Pratt, 2008). As we learn more about students’ mathematical behaviour in such dynamic environments, these strategies should be refined and enriched with empirical evidence. Our
teaching experiments with pre-service student teachers have been successful, and we invite
classroom teachers at all levels to try these pedagogical tools in their teaching practice and build
students’ mathematical competence by providing them with active mathematical experiences
and especially mathematical modeling activities (Lesh & Doerr, 2003).

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GEOGEBRA AS A PEDAGOGICAL TOOL: A PRELIMINARY TAXONOMY

Lingguo Bu1, Frackson Mumba2, Yazan Alghazo3
1,2,3Southern Illinois University, Carbondale, Illinois, USA.
1lgbu@siu.edu, 2frackson@siu.edu, 3alghazo@siu.edu

Abstract
GeoGebra provides a resourceful toolkit for mathematics educators to reflect on their teaching practice and further develop new ways to connect, extend, and enrich their instructional activities. Under the theoretical framework of model-centered learning, we reflect on our field experience with GeoGebra-integrated mathematics teacher development and propose a preliminary taxonomy for the pedagogical uses of GeoGebra in mathematics teacher education. We use a specific modeling problem as an example.

Keywords: Pedagogical reflection, instructional design, teacher education

Introduction
GeoGebra (www.geogebra.org) provides an accessible mathematics learning environment that integrates multiple mathematical representations, computational utilities, and documentation tools. Within GeoGebra, not only are the various representations automatically linked, they can also be dynamically manipulated to illustrate or simulate the dynamic nature of mathematical ideas. In alignment with our growing knowledge about mathematical understanding and its complexity in terms of multiple representations and dynamic mental models (Carpenter & Lehrer, 1999; Goldin, 2003; Hiebert & Carpenter, 1992; Moreno-Armella, Hegedus, & Kaput, 2008; Seel, 2003), GeoGebra stands as an equitably accessible digital environment that appeals to mathematics educators and students at all levels as they strive to promote mathematical sense-making and deep understanding. Furthermore, from a tool use perspective (Vygotsky, 1978), while GeoGebra is, in most cases, used initially by mathematics educators as a technical tool to support teaching and learning, it gradually evolves, first, into an psychological tool or an instrument that facilitates a teacher’ instructional plans and strategies, and, further, into a pedagogical tool that facilitates a teacher’ classroom practice in many aspects of mathematics teaching. In this paper, we reflect on our own experience with the integration of GeoGebra in both preservice mathematics teacher education and inservice professional development programs across a period of three years. We assume the role of teacher educators, upholding the position that teacher educators themselves are reflective learners in both preservice and inservice programs. We believe that the emerging GeoGebra user community is indeed a group mathematics
teachers and students who are not only actively inventing and experimenting new ways of mathematics teaching, but are themselves learning or relearning about the mathematics. As Freire argues convincingly:

There is, in fact, no teaching without learning. One requires the other. . . . Whoever teaches learns in the acting of teaching, and whoever learns teaches in the act of learning. (Freire, 1998, p. 31)

As a focus, we discuss the pedagogical roles of GeoGebra in our preservice and professional development field work. The term pedagogy refers to the ways of teaching under a certain theoretical framework for teaching and learning. In what follows, we present an overview of a model-centered framework for teaching mathematics using GeoGebra, followed by an illustrative example and a tentative taxonomy of the pedagogical roles of GeoGebra.

**Theoretical Framework**

To foster mathematics teachers’ growth in teaching mathematics with new digital technologies, we situated our instructional design and its enactment in the theoretical framework of multiple representations (Goldin, 2003) and Model-Facilitated Learning (MFL) (Milrad, Spector, & Davidsen, 2003). The theory of multiple representations has served as the foundation for the reform-based conception of mathematical understanding (Hiebert & Carpenter, 1992). A mathematical representation refers to both the cognitive process and the external product of mathematical reasoning, taking on such forms as graphs, algebraic expressions, and various informal diagrams or tables. From the learning perspective, the use of multiple representations contributes to the resolution of intrinsic ambiguities within the representation system (Goldin, 2003), leading to the ultimate reification of a mathematical concept as a mathematical object (Sfard, 1991, 1994).

Furthermore, recent conceptions of mathematical understanding have placed much emphasis on a learner’s ability to use and navigate through multiple representations, which, theoretically speaking, indicates a learner’s knowledge of the mathematical processes and the corresponding conceptual structure in the form of dynamic conceptual and mental models (Doerr & Lesh, 2003; Gravemeijer, 2008; Seel, 2003).

However, effective instructional design in mathematics requires more than multiple representations. In developing instructional sequences, we were further informed by the MFL framework (Milrad et al., 2003), which seeks to promote meaningful learning and deep understanding by fostering learners’ development of a holistic view of a complex problem situation. The MFL framework consists of modeling tools, multiple representations, and system dynamics methods that allow learners to build models and/or interact with existing models as part of their effort to understand the structure and the dynamics of a problem situation. MFL recommends that learning be situated in a sequence of activities of graduated complexity,
progressing from concrete manipulations to abstract representations while learners are engaged in increasingly complex problem solving. Through the use of multiple representational tools, MFL further maintains the transparency of the underlying models that drives the behavior of a system simulation.

A Mathematical Example

Worthwhile mathematical tasks aligned with elementary (K-8) mathematical standards are at the core of our GeoGebra-integrated courses and programs. In selecting or designing learning tasks, we seek open-ended realistic problems that are familiar to our teacher participants and yet provide unexpected solutions or cognitive conflicts. Many problems are directly from or reviewed by professional mathematicians. Our overarching goal is to challenge the traditional views of mathematics held by the majority of our participants and help them develop insights into mathematical problem solving by engaging them in non-trivial mathematical problem solving and modeling. In most cases, we expect our teacher participants to develop profound understandings of elementary mathematical ideas. In accordance with the Dynamic Principle in teacher development, we believe that “the knowledge that teachers need should move from understanding relationships that are static to those which are dynamic” (Doerr & Lesh, 2003, p. 135). Mathematical examples include problems such as: a) Given the diagonal of a rectangle, how could you rebuild a (or the) rectangle? b) If the class measures 20 circular shapes of various sizes, how could you make sense of the relationships between the circumference and the diameter? c) A circle is given with no indication of the center, how can you locate its center? All the problems call for mathematical concept play (e.g., Davis, 2008). In the rectangle problem, for instance, our preservice teachers came with an unconventional definition of a rectangle: a quadrilateral whose two diagonals bisect each other and are congruent. One preservice teacher, in fact, imagined rotating the given diagonal around its midpoint to obtain a rectangle, thus relating a rectangle to a circle. Such ideas can be readily implemented with GeoGebra with minimal instructor guidance. There are, indeed, infinitely many such rectangles. As a focus point, we invite our readers to think about following problem and how GeoGebra may change or enhance their instructional strategies and practice when the problem is used with a group of elementary (K-8) mathematics teachers.

As shown in Figure 1, some treasure was buried on the beach at a location determined by a palm tree and a boulder. To find it, start from the palm tree (P), walk toward the sea for some distance to some point W. Mark W. Then, go back to the palm tree, at a right angle to PW (clockwise), walk the same distance toward land and mark a point X. Then, go to the boulder (B), at a right angle
to BW (counter-clockwise), walk a distance of BW to the land, mark a point Y. Find the mid-point T between X and Y, which is location of the treasure!

Figure 1. A treasure hunting problem.

**Instructional Reflections on the Pedagogical Roles of GeoGebra**

Being experienced users of GeoGebra, the instructors are familiar with the potentials of GeoGebra for teaching the Treasure Problem in the classroom. First, we note that the primary goal of this task is for our participants to experience genuine mathematical problem solving and what it means to teach mathematical problem solving. Second, the problem can be solved in a paper-and-pencil approach, which, of course, has its limitations in addressing what-if and what-if-not questions (Brown & Walter, 2005). In our practice, we found that it is indeed helpful to have participants try to solve the problem on paper as a way to understand or even question the problem and to identify key ideas involved in the problem.

When GeoGebra is brought into the teaching process, however, a variety of pedagogical possibilities arise. First, from students’ perspective, students can be engaged in mathematical modeling, problem exploration, and open-ended questioning. They can diagnose their working model(s) for the problem and make adjustments. They can also claim ownership on their mathematical construction, including, in most cases, the good mistakes that are naturally part of learning mathematics.

Second, from a task perspective, the problem takes on new dimensions when GeoGebra is used in instruction. The problem itself becomes an example from a large, and, in fact, infinite collection or problem space of similar problems. At many stages of problem solving, there exist alternative pathways. For instance, to make a 90-degree turn and walk the same distance could be accomplished by a) the 90-degree rotation of a segment, or b) the construction of a 90-degree angle followed by a segment of given length, or c) using perpendicular lines and circles (see Figure 2). The whole problem could be used to invent another problem: If I were to place the treasure for others to hunt, how can I find its proper location?
Third, from a teacher educator’s perspective, the instructor can design partially completed worksheets to scaffold and accommodate the diverse needs of students and allow various entry points to the problem. GeoGebra is also a tool of assessment, allowing the teacher to look into student teachers’ processes of thinking. More importantly, the instructor can create his or her own instructional materials in support of his/her own reflection and participation in the local and global community of mathematics educators. For an example, in the treasure problem, the two right angles or one of them could be completed for certain students who may have difficulty coordinating the complex relations in the problem.

Fourth, GeoGebra enhances the learning environment with its multiple representations, computational utilities, documenting tools, and web-friendly features that extend the scope of teaching and learning beyond the walls of the classroom. In the treasure problem, both students and instructors can share their dynamic constructions online through the Blackboard system or elsewhere. Such dynamic solutions can be further demonstrated, exported, or modified in support of other learning objectives. Figure 3 shows the underlying mathematical structure of the treasure problem and where the treasure should be buried to set up the problem. The dynamic construction can further be used to make conjectures about the problem, the properties of midpoint T, and congruent triangles (Figure 3). A formal proof may be the next step if so desired.

Based on the above analysis and similar cases in our use of GeoGebra in teacher preparation and professional development programs, we classify the pedagogical roles of GeoGebra under the theory of model-centered learning and instruction (Table 1) as a way to make sense of the instructional uses of GeoGebra and to generate new ideas for teaching mathematics teachers to use technology.
Figure 2. A GeoGebra model for the treasure hunting problem.

Figure 3. Where should the treasure be buried if one were to set up the problem? The GeoGebra model shows that when W moves, the square and point T do not move at all.
### Table 1. A Taxonomy of the Pedagogical Roles of GeoGebra in Mathematics Teacher Education

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Pedagogical Roles of GeoGebra</th>
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| **Students** | Engaging students in exploring, asking what-if and what-if-not questions  
Supporting students’ decision-making  
making confirmations, conjectures, drawing conclusions, tweaking problems,  
or inventing new problems  
Diagnosing students’ working models for math ideas  
revealing their thinking  
identifying weak points in knowledge  
pointing directions for improvement  
Promoting ownership of mathematics learning/construction/reflection |
| **Tasks** | Extending the problem/example space  
changing initial conditions  
finding singular or special cases  
Enabling alternative pathways to problem solving  
model-based solutions  
informal arguments, formal proofs |
| **Educators** | Supporting reflections on the part of the teacher and teacher educators  
action, reflection, autonomy, networking (Llinares & Krainer, 2006)  
Scaffolding to accommodate diverse needs and entry points  
Networking with the local and international community  
using existing resources  
active contribution  
Relearning about mathematics and present-day students and (student) teachers  
Promoting ownership of instructional sequences, lessons, and artifacts  
Assessing students’ understanding of mathematics  
model-based assessment  
focusing on the processes of problem solving |
| **Environment** | Providing multiple representations and simulations  
Providing computational utilities  
Providing documenting and organizing utilities  
graphs, images  
dynamic worksheets  
intermediate learning objects  
interactive whiteboard  
Reaching beyond the space and time of the class  
teacher reflection  
student reflection  
artifacts for documentation & research |
Conclusion

GeoGebra provides a rich set of resources for teacher educators to enhance and reflection on their practice in both preservice and inservice mathematics teacher development. In proposing a tentative taxonomy of the roles of GeoGebra as a pedagogical tool, we are informed by the basic principles of model-centered learning and instruction, aiming to critically reflect on our own teaching experience and our use of new dynamic learning technologies in teacher education. As Shulman (2002) put it, in making the difference, we have tried to make sense of our own experience. Indeed, as teacher educators who learned much of the mathematics decades ago, we were relearning or even diagnosing our own understanding of the mathematics we think we have confidence in teaching. GeoGebra stands as a tool for thought and reflection for mathematics teachers educators in terms of student learning, teacher learning, and our own learning as the world changes and brings us new tools, new opportunities, and new challenges. In closing, we quote Shulman’s view about design and invite all mathematics educators and technology enthusiast to reflect on GeoGebra-based mathematical modeling and instructional design: “Design is a matter of exercising understanding, as well as applying skills, under a variety of constraints and contingencies” (p. 41).

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Students entering post-secondary mathematics courses may be considered ‘marginalized’ for multiple reasons: previous credentials do not meet entry standards, cultural/language barriers prevent deep understanding, length of absence from formal education, lack of self-confidence or early abandonment due to rigour of subject matter. Without a strong framework to build from, students who are weak in mathematics may be at a significant disadvantage if career interests lie in applied science or engineering technology fields. Instead of mathematics becoming a gatekeeper subject, many higher education institutions have developed fundamental courses to provide a pathway for students to strengthen their understanding and confidence while enhancing communication, critical thinking and problem solving skills.

Many instructors teach mathematics in the same way they were taught. Traditional methodology has classroom time spent writing problems on the board, students copying into notebooks, limited interaction/class discussion, followed by textbook homework. Foundational teams seek to provide an innovative approach to the teaching and learning of mathematics – a methodology that more closely resonates with the way students of today prefer to learn. At the college studied, the aim is to increase the integration of technology into all aspects of teaching mathematics and to develop a student-centered community learning model.

As this common foundational mathematics course is taught across schools ranging from applied science to electrical engineering technology, a compilation of notes that can be made available to students through a learning management system (LMS) was required. These e-notes represent a summation of the wealth of knowledge available from the multiple disciplines and experiences of both new and seasoned teachers. This material provided the scaffolding which individual instructors tailored to suit the technical requirements of their discipline while at the same time ensuring all students received a common level of mathematics understanding. Prior to the class,
the notes were posted to the LMS, which students downloaded and brought with them. Instead of the traditional lecture style of content transfer, more in-class time was devoted to discussion and determination of comprehension. Although a valuable first step, this asynchronous transfer of information only provided a more efficient mechanism of delivering content. While this methodology required students to be more responsible for pre-class preparation, the additional budget strain of printing notes became prohibitive for some students. Further, it fell short of the goals of the teaching team to increase student engagement, enhance student learning and ability to apply concepts, and to demonstrate an innovative teaching process throughout the college/math/technology/outreach community.

In the fall of 2008, the team received a Hewlett-Packard Higher Education Technology for Teaching Grant. The grant provided 21 pen-based tablet pc’s, sufficient to develop a pilot research study to investigate use in the classroom. The advantage of having this technology for teaching and learning mathematics was immediately realized. The pen (stylus) allows teachers and students to write math formulations in a familiar way, in contrast to the keyboard restrictions that are encountered when typing on a conventional laptop/desktop device. Aside from some initial awkwardness of learning to write on a screen, students adapted quickly, and soon realized the annotation advantages, including erasing, that an electronic pen provides.

The team determined that connection between teacher and student tablets was required and DyKnow software licenses were purchased by the college. DyKnow software is compatible with either Word document or Power Point presentations. Teachers load their personalized teaching documents into the DyKnow Writer and students log into an open session. With the capability of releasing one or multiple slides (panels) at a time, teachers give students the ability to work at their own pace. Students save their session to a college server which they can access via the internet at any time. For those students without internet connection in the home, notes can be accessed from college libraries or computer laboratories.

The use of the tablet pc and collaborative software has resulted in an innovation in teaching methodology. Lectures now take the structure of: 1) interactive focus activity, 2) concept development, 3) collaborative practice, 4) application and 5) reflection of learning. As some lag time occurs while students log in, the instructor often begins with an interactive focus activity. By projecting onto the screen and using the link feature of the collaborative software, students are directed to an internet learning object/applet/video and are encouraged to investigate individually, in pairs or in groups, then discuss findings. With the internet at their fingertips, they will often go to a search engine to seek meaning of unfamiliar terms. The embedded link on the panel becomes a part of the class notes which students, using their personal computer, can open.
at a later time for more detailed analysis or review as necessary. Once all students are logged in, the instructor can un-tether themselves from the projector and freely move about the room with their workspace on their arm – as can students in a mobile/wireless environment. Open-ended questioning and class collaboration enhances the learning experience derived from the focus activity. By capturing this concept development on screen, all students have this learning experience recorded as part of their notes. Without having to be concerned about copying notes correctly, students can focus on the teacher’s process used for solving, add notes that arise from discussion and annotate with pen colours and highlighter to provide focus on key learning.

Multiple features of the DyKnow software are employed to ensure that practice is collaborative and learning outcomes are met in unique and interesting ways. Students are placed into online groups so answers that are difficult to achieve independently can be developed through cooperation. The group can submit this panel to the teacher who then adds it to the class notes, where it can be viewed on individual tablet screens. In addition, the teacher can ‘share control’ of the session. While sitting at their own tablet, single/groups/all students can be selected to demonstrate their approach to problem solving by being given control of the class screen.

Students comment that this functionality allows them the opportunity of becoming the temporary teacher, requiring them to take greater responsibility for their own work. They quickly understand why an answer that uses poor form/illegible writing or skips steps is not desirable to others viewing their work. Students comment that an answer with correct process increases their pride in their work, builds their confidence and makes them more likely to continue their efforts outside of the classroom. To enhance the learning benefits of this feature, teachers provide panels with tables that have questions duplicated in two columns. Students are instructed to work independently on one side to attempt personal solutions. After an appropriate time, share control is granted to the entire class and students ink in solutions to the question of their choice. Once access is closed, the class works together to discuss and modify solutions. As all responses are anonymous, students gain this understanding without embarrassment. Of most importance, students compare their personal work to others and begin to understand the necessity of self-correcting habitual errors. The ability to ‘send panels’ to the instructor allows for a backchannel conversation to occur with students that prefer to have their teacher review their solutions on a regular basis. The panels are viewed at a quiet moment in class to provide immediate feedback, or saved to return directly to individual students outside of class time. The send panels feature is also used to share application questions with fellow classmates. Students are asked to make-up questions that are of interest to their own experiences, solve them, and send them to the instructor. These new panels are collected and appended to class notes, providing multiple examples of solved problems for students to review. By using embedded links to applets/learning objects/videos/Web Quests, teachers provide an
application experience which results in a clear understanding of how the learned math skill will be used in future discipline-specific situations. Reflection of learning outcomes is evaluated throughout the lecture by using multiple DyKnow tools. For example, the teacher can ‘request status’ of students which asks them to indicate their level of understanding by sending a red, yellow or green stoplight. Polling students quickly determines comprehension and if required, additional panels are inserted to supplement the discussion. These methodologies are used to either elicit conversation or develop reflective practice. Further, students are asked to submit reflection panels as part of an online journal which teachers review and return. This online, collaborative workspace is then saved to the college server, and becomes a part of the students’ virtual binder.

The combination of the tablet pc’s and collaborative software provides a workspace for students to engage in two-way synchronous communication. This learner-centered environment results in student perception of enhanced engagement, supported by increased attendance and retention in the course. Survey data and student focus group comments: “…it isn’t like any other class when you sit there and just listening to a teacher speak on and on…you’re actually interacting with others.”, and “…tablet…gave you an opportunity to participate…without the embarrassment of public speaking” or “my participation…have grown significantly because of the confidence it brings when im comfortable of where I am.” Help teachers to understand the benefits of the development of a personal learning community and its impact on student confidence and understanding of the subject material. Student performance increased measurably and comments: “Makes me perform better since it is a new and interesting way to approach classes, I tend to pay attention more and thus doing better in work and tests”, “…I actually did my work and it was done right and organized…” and “…It made learning…much easier, more enjoyable and improved my performance…” allow teachers to conclude that this active learning environment results in improved student understanding and encouragement of necessary critical thinking skills. “The tablets were useful because I can see what the teacher is writing, students can share answers anonymously, we can correct each other’s work”.

With a greater emphasis placed on class development of notes, students are required to take responsibility for their learning and thus determine for themselves where weaknesses and error occur. Faculty must be amenable for adaptation in teaching philosophy and a shift in thinking because students determine the pace of classroom delivery and are given the opportunity to contribute to the teaching role.

Further advances at the college studied include the opening of two additional tablet classrooms. One laboratory seats 40 students and addresses the technological implications of larger class sizes
that are reflective of a more standard class enrollment. The second is located at another campus where students are pursuing careers in different fields of technology. Its purpose is to determine if deployment of this teaching methodology can be expanded to other disciplines, in particular where gender may have an influence.
GEOMETRICAL PROOFS, BASIC GEOMETRIC CONFIGURATIONS AND DYNAMIC GEOMETRY SOFTWARE

Margo Kondratieva
Memorial University, Canada

Abstract

Knowing geometrical facts and understanding their proofs are of equal importance for learning geometry. One way to target both goals is to study basic geometric configurations (BGCs). BGC is a geometrical figure that depicts a statement along with auxiliary elements pertinent to its proof. Benefits of using BGCs in teaching geometry can be enhanced by employment of applets produced with a dynamic geometry software. We discuss innovative practices that apply this idea to teaching proofs at the university level. A sample classroom scenario illustrating our approach is presented. Results of a survey of 13 students who had taken the course highlight a potential of this practice and suggest directions for further research.

Key words/ themes: promoting conceptual understanding of mathematics through explorations, visual proof, computer technologies for learning/doing mathematics.

Motivation and overview of our innovative practices

This paper concerns innovative practices and development activities in teaching Euclidean Geometry at the undergraduate university level. As a consequence of the current grade school mathematics curriculum, students enter this course with a limited amount of geometrical knowledge often restricted to a list of formulas and facts for certain geometrical figures such as the triangle and circle. Several students have some familiarity with formal deduction (e.g. two column proofs), but they often lack understanding of the meaning of this process. Thus the university course aims to give students a richer background and experiences in geometry as well as allowing students to understand the essence of proving practice. Learning to proof is vital for learning mathematics (Rav 1999; Balacheff 2010). First, proof as a means of validation, reinforces a precise and highly logical way of thinking based on axioms, definitions, and statements, which link and describe the properties of mathematical objects. Second, proofs include mathematical methods, concepts, and strategies also applicable in problem solving situations (Hanna & Barbeau 2010). Despite their central role in mathematics proofs often receive insufficient appreciation and epistemological understanding from students, who rely on empirical evidence rather than on formal deductions of mathematical theorems (Coe & Ruthven 1994). “Pupils fail to appreciate the critical distinction between empirical and deductive arguments and in general show a preference for the use of empirical argument over deductive reasoning.” “Proof is not
used’ as a part of problem-solving and is widely regarded as an irrelevant, ‘added-on’ activity” (Hoyles & Jones 1998, p. 121).

This state of affairs identifies the needs for “mathematical activities that could facilitate the learning of mathematical proof” (Balacheff 2010, p. 133) and “problem situations calling for an interaction between visual methods and geometrical methods” (Laborde 1998, p.114). One possible approach “is centered around the idea that inventing hypotheses and testing their consequences is more productive than forming elaborate chains of deductions” (Jahnke 2007, p.79). The process of making conjectures and inventing hypotheses requires mathematical intuition, which develops through students’ experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers 1990). At the same time, “current curricular trends, promulgating proving processes based on experimentation and conjectures, will lead to an effective learning of proof, with proof attaining its full meaning in the learners’ understanding only if these processes are set within a genuine process of building ‘small theories’… From these axioms/hypotheses would be elaborated hypothetico-deductive networks, which would then be confronted with the initial experimentations and conjectures” (Tanguay & Grenier 2010, p.41). A productive way of incorporating experimentation and proving needs to be found so that “proofs do not replace measurements but make them more intelligent” (Jahnke 2007, p.83). This is possible due to several roles (besides validation statements) that proofs may play in mathematical thinking (Hanna 2000; De Villiers 1990). First, at the informal deduction stage, proof as explanation of empirical observations is most appropriate. Next, students “should build a small network of theorems based on empirical evidence” and become accustomed to “hypothetico-deductive method which is fundamental for scientific thinking” (Jahnke 2007, p.83).

At this stage, the proof functions as a “systematization (the organization of various results into a deductive system)” and “construction of an empirical theory”, preparing students for rigorous proofs aiming at establishing truth by deduction or “incorporation of well-known facts into a new framework” (Hanna 2000, p. 8).

My approach to teaching Euclidean Geometry emphasizes the use of basic geometric configurations (BGCs) - fundamental geometric facts expressed in drawing (Kondratieva 2011). Such drawings contain auxiliary elements and labels (equal angles, equal segments, perpendicular and parallel lines) that allow remembering the statements along with the ideas of their proofs. BGCs are the stepping stones to proving or solving geometric problems. But figures support visual thinking only if a learner grasps the mathematical structure they represent (Arnheim 1969). This calls for the following teacher’s actions: (1) Asking students to explain what relations they observe in a figure and what they think about the role of the auxiliary lines drawn on the original figure. (2) Constantly showing connections to already learned geometrical facts and
focusing students' attention on the key ideas used in a particular solution. (3) Demonstrating several proofs or solutions of the same problem in order to show connections between geometry, trigonometry and algebra. (4) Directing students’ attention to the implications, converse and equivalence of statements. (5) Helping students summarize their findings in the form of a mathematical statement. (6) Surprising students with an unexpected conclusion or asking them to correct errors in a flawed reasoning (Kondratieva 2009, 2011).

Taking into account that “computers can offer a new context for designing innovative activities to address the main problem of linkage between empirical experiments and deductive reasoning” (Osta 1998, p 111), I introduce my students to dynamic drawings (applets) produced in GeoGebra (GG). Several types of tasks helping to connect visual evidence and geometrical facts can be considered: (i) moving from a verbal description on a geometrical figure to a drawing; (ii) explaining the behavior of drawings in geometrical terms (interpreting or predicting); and (iii) reproducing a drawing or transforming a drawing (Laborde 1998, p. 115). The students employ my GG applets and create their own drawings that help them to understand and interpret BGCs. Many BGCs allow “dynamical visual proofs, which are based on ‘drawing in movement’ that can be properly performed in a dynamical environment” (Gravina 2008).

The novelty of this approach consists of combining the methodology of the BGC approach with the advantages offered by the geometry software, in order to balance empirical and deductive practices. First, students read and analyze sample proofs and identify BGCs and key ideas pertinent to the proofs. At the same time students construct interactive applets in GG with the requirement to make the constraints described in the statement indestructible by dragging. This forces them to use geometrical properties of the object they draw. Students are asked to show auxiliary lines and measurements pertinent to the idea of the proofs. Students are encouraged to invent alternative proofs to the statements they analyze and interpret with help of GG. Students are given examples of all these activities in class. They discuss BGCs with their teacher using both static and dynamic drawings. As the semester evolves, the students are provided with fewer hints for problems and are asked to continue building GG applets and experiment with them in order to find their own solutions. In this way students gradually adopt the Euclidean (synthetic) geometry tradition of proofs and learn to recognize and apply BGCs. The students learn to observe and explain individual empirical facts, then build, and check their “small theories” based on many dynamic and static drawings.
The six point circle theorem in the classroom

In this section we illustrate our approach with a concrete example of a study of the six point circle theorem. This theorem states that: "For a triangle ABC with midpoints on the sides L, M, and N and the feet of the altitudes D, E, and F, all lie on a circle."

Below we give a sample dialog between a teacher and a small class of undergraduate students equipped with dynamic geometry software (DGS).

Teacher: Good morning, class! Do you know how many circles one can draw given 3 arbitrary points?

Anna: I think only one. Yes, you can always draw a circle through 3 points, but only one.

Teacher: Everybody agree?

Nick: I agree. But the points must not be collinear; otherwise you can’t draw a circle.

Anna: Oh, I forgot this exception. But other than that…

Teacher: Can you explain why?

John: One can construct the center of such a circle and thus find the circle. We did it last time. It’s the intersection of perpendicular bisectors of the sides of the triangle with vertices at given 3 points. This is a basic geometric configuration we constructed at home in Geogebra (see Figure 1).

Matthew: I also noticed that Geogebra has a build-in function “Circle through Three Points” that allows drawing a circle by 3 points.

Teacher: Very good. I hope everyone made the applet and recalls this fact now. Today we are going to experiment with other construction. Please draw a triangle ABC. Find the midpoints of the sides and call them L, M, N. Draw a circle through L, M and N. Draw an altitude CF with foot F on AB. Drag vertices of the triangle. What do you notice?
Figure 1: Basic geometric configuration showing that circum-center $O$ lies at the intersection of perpendicular bisectors of the sides of a triangle.

Students follow teacher’s instruction and use various build-in functions such as “Midpoint or Center”, “Perpendicular Line”, “Circle through Three Points” to create their applets. Students are required to obey all given constrains and then experiment with their drawings and report their observations.

Kelly: I think point $F$ always lies on the circle defined by the midpoints $L$, $M$, and $N$.

Teacher: It looks like that, indeed. Now we have to either explain this fact or find a situation where this is no longer true.

The students play with their applets for some time. Often, students quickly see the fact and get more and more convinced by viewing results of dragging points. At the same time they may have difficulties to find an explanation of the observed geometrical behavior. When Teacher notices that no reasonable explanation emerges from this activity, she decides to give students a hint in the form of her own applet, prepared for this lesson. This applet embraces a BGC because it represents a statement and an idea of its proof. The goal of the teacher is to guide the students letting them to see this idea and articulate it in their own words.

Teacher: Please take a look at my drawing (Figure 2). What do you observe?

Mathew: I see a quadrilateral with vertices at the points of interest. Should we try to show that this quadrilateral is always cyclic?

Teacher: Perhaps. What can you say specifically about this quadrilateral?
Nick: It is a trapezoid. The line MN, connecting midpoints of the two sides of the triangle, is parallel to the third side of this triangle.

Teacher: Yes, good. What do we need to prove now about this trapezoid if we had to show that it is cyclic?

Kelly: To be cyclic it needs to be isosceles. But is it?

Teacher: Let’s look at the figure and find out.

Figure 2: The six point circle theorem: case of acute triangle.

John: Since segment LN is a midline in triangle ABC, its length is a half of length of the side AC. Now what about MF? On your hint-figure it looks like MF is a radius of the circle with diameter AC. If this is the case, then we are done.

Nick: Right. Point F is the foot of the altitude CF, so CFA is 90 degrees angle and thus the hypotenuse CA is a diameter of the circum-circle. This is a BGC we recently learned. (See Figure 3.)
Figure 3: Basic geometric configuration showing that hypotenuse of a right triangle is a diameter of the circum-circle.

**Teacher:** Very good. Let us summarize. What is the key idea of the proof? What are BGCs required in completion of this proof? Please write down your proof and make sure that your statements do not contradict the observations made with the applet. What difference do you observe in the proof for the cases of acute and obtuse triangles?

Students drag points on the applet to make the triangle obtuse and observe that all elements of the proof remain the same (Figure 4).

Figure 4: The six point circle theorem: case of obtuse triangle.

**Teacher:** As a matter of fact, we just proved so called the six point circle theorem. Why does it refer to six points?
Anna: We proved that three side midpoints and foot of an altitude lie on a circle. But we can repeat this for other altitudes in the triangle and will get the foot of each of them on this circle defined by the midpoints. I can show you this on my applet.

Teacher: Very good. Next time we will study the nine point theorem, which is an extension of today’s discussion, so please save the applet you made today for the next class.

Conclusions

The conversation given in the previous section exemplifies our innovative practices in teaching geometry. Its main component is a guided discussion of properties and details presented in a prefabricated dynamic figure. All negotiations and experimentations with the applet aim at helping students to form a visual mental image of a geometrical statement along with an argument supporting this statement.

A survey of 13 students taking the Euclidean Geometry course in the Fall of 2010, reveals the following results (in parentheses the number of positive responses is shown) : “I am learning a lot of new things about geometry in this course” (13); “I am developing my understanding of and ability to proof to a higher degree” (12); “The use of GeoGebra helps me to make a connection of symbolic and visual representation” (12) and “to interpret and understand theorems in geometry” (11); “GG is aesthetically pleasant” (11), “insightful” (12), “helping to activate my knowledge”(11); “GG encourages me to make and test conjectures”(11), “facilitates exactness of my mathematical thinking”(11); “GG allows me to try a larger range of possibilities compare to pen and paper approach” (13).

The survey results confirm a potential usefulness of the proposed approach and concur with the observation that experimentation and conjecturing in dynamic geometry environment must be done in conjunction with other activities (e.g. analysis of BGCs and deductive proofs) that help students to build their networks of theoretical knowledge. Further research is required in order to highlight details and nuances of synchronization of the heuristic and logical components of students’ work within the innovative practice described in this paper.

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FINDING THE AREA OF A CIRCLE: AFFORDANCES AND DESIGN ISSUES WITH DIFFERENT IGS PROGRAMS

Kate Mackrell
Institute of Education, University of London, U.K.
kmackrell@ioe.ac.uk

Abstract

In order to consider the potential impact of differences in program affordance and design on student learning, an exploratory inspection involving a series of tasks related to finding a connection between the radius and area of a circle was performed using Cabri II Plus, Cinderella, GeoGebra and Geometer's Sketchpad. These tasks involved different aspects of number, algebra, geometry, and handling data. It was found that although all programs had sufficient affordance for the tasks, there were significant differences in the way such affordances were implemented and a number of particularly problematic features of GeoGebra were identified, together with their potential impact on student learning.

Introduction

Laborde (2007), in a summary of the research on the use of interactive geometry software (IGS) in the classroom, comments on the increasing awareness that learning outcomes depend not only upon tasks and pedagogical context, but also on software design. Jackiw (Butler, Jackiw, Laborde, Lagrange & Yerushalmy, 2009) has called for more specific attention to be given to the detail of design decisions in IGS and has identified a number of important design decisions. Further design decisions have been described by Laborde & Laborde, 2008 and Richter-Gebert & Kortenkamp, 2010. However, there is currently little discussion of design decisions and issues in GeoGebra.

Mackrell (2011a) created a categorization of the operations possible in an IGS in which to situate design decisions and, using Cabri II Plus v1.4.3 (2008), Cabri 3D v2.1.2 (2007), Geometer’s Sketchpad (GSP) v5.01 (2009) Cinderella v2.1 (2010) and GeoGebra v3.2.46 (2010) explored aspects of three fundamental operations: construction, object (including dragging and redefining) and changing the view (spatially or semantically), showing that there is a great diversity in both the design decisions that need to be made and in the decisions made by individual designers. Mackrell (2011b) considered the impact of particular design decisions on the potential for the integration of number, algebra and geometry and concluded that particular design decisions were likely to either facilitate or impede the development of student understanding in these areas, and called for further research on student responses to conclusively evaluate the impact of such design decisions.
Although none of the programs explored were without problems, GeoGebra was found to have made a number of particularly problematic design decisions. This paper focuses on these decisions, comparing them to those made in other IGS programs, and discusses their potential consequences.

Method

IGS programs are highly sophisticated, with a vast range of affordances. Accordingly it was decided to take a “snapshot” of the capabilities of each program by performing an integrated series of tasks with a particular curriculum goal that involved a range of program affordances. The tasks had the aim of finding and verifying a formula giving the area of a circle in terms of its radius. The topic, finding the area of a circle, is universal, and each task involved processes that would also apply to other mathematical explorations. The pedagogical approach (involving exploring and gathering information about a mathematical situation, making and testing conjectures, then generalizing and proving results) has been promoted extensively in the UK since the 1980’s.

Results

Task 1: Create a circle, and a segment to represent its radius

In each program, a circle was created given a centre point and a point on its circumference. This tool was the simplest tool in common to all the programs.

The design decisions involved at this stage are which tools to make available and the way in which these tools are to be used. In considering the available circle tools in each program, it is useful to distinguish between two types of tool functionality. A tool has atomic functionality if it transforms input in a way which cannot be achieved by any other tool or sequence of tools. For example, the tool which creates a circle by center and radius point has atomic functionality. A tool has molecular functionality if its output can be produced from the given input by an operation or series of operations using tools with atomic functionality. For example, the tool which creates a circle given a centre point and a segment for radius length has molecular functionality, as it may be replaced by first using the atomic tool which measures the length of a segment and then by using the atomic tool which constructs a circle given a centre and numerical radius (Mackrell, 2011a). The functionality of a program as a whole depends upon its atomic tools, particularly when new custom tools may be created to give tools with the required molecular functionality. The three atomic functionalities represented among the five programs include the two mentioned in the examples above and also the ability to create a circle given its centre point and selecting an initial value for its radius by clicking on the screen. This circle has the advantage that dragging the centre point moves the circle, and dragging on the circumference changes the
radius, giving a very simple representation of a circle to explore, dependent on no other objects. Only Cinderella and Cabri II Plus have this functionality: of the five tools to create a circle in GeoGebra, only two are atomic.

However, there are also a number of reasons for making molecular tools available: the complexity of performing an operation with only atomic tools, the encapsulation of fundamental relationships and constructions, the introduction of appropriate mathematical language and to enhance the perceived affordance of a program. There are also problems, however, in that introducing further tools adds complication to the interface. and hence it is important to be able to control the number of tools available. All of the programs allow custom tools to be created, but only Cabri II Plus and GeoGebra allow the suppression of tools.

In using a tool, a fundamental design decision is whether the user needs to choose the tool (action) first, or the objects to which the tool is applied first. The first method is called A/O (for action-object) and is used in almost all operations with all but GSP. The second method is O/A, used for a large number of GSP operations. The importance of A/O is that it enables a large number of other features, such as points-on-the-fly, in which points may be created in the course of using a tool. Cinderella does not allow consistently allow points on the fly, but the other programs allow these in all A/O operations. A/O also enables a description to be given once a tool is chosen and as objects are selected. An advantage to the tool help in Cabri is that this can be turned off when no longer necessary. Tooltips and highlighting of selectable objects are also enabled. GeoGebra’s tooltips consist of identifying all objects rather than just objects relevant to the construction, however. Highlighting in GeoGebra is also less noticeable than in the other programs: GSP is particularly clear in its O/A operations.

A/O and O/A/O selection also enable graphic previews, in which, as soon as enough objects are selected, a preview is given of the object that will result. An example is the construction of a polygon, in which all programs give a preview of the polygon as vertices are selected. However, Cinderella and GSP give a graphic preview of points to be created on objects, but GeoGebra and Cabri II Plus do not. GeoGebra and Cinderella also do not give a graphic preview for a conic through five points.

**Task 2: Measure the area and radius of the circle**

Screenshots of the figures constructed are shown below.
Two areas in which design decisions are necessary can be seen in the screenshots. One of these is the calculation and display of measurements. The Cabris and GSP initially give units in cm or degrees as appropriate and GeoGebra and Cinderella give no units; different units may be chosen for the Cabris, GSP, and Cinderella, but in GeoGebra, measurement units can only be displayed on axes rather than on individual measurements. This is understandable, given the underlying algebraic representation used in GeoGebra, but problematic in teaching.

One particular design decision illustrates the impact of differences in mathematical convention. When a number is rounded to a certain number of decimal places, the convention in German-speaking countries is to leave off all trailing zeroes. Hence, in Cinderella and GeoGebra, 3.004 rounded to two decimal places is represented as 3. In contrast, GSP and the Cabris represent the number as 3.00: 3 would represent the integer 3 or the number rounded to the nearest whole number. A statement such as “3*4 = 12.02”, where all numbers are represented to the same degree of accuracy, is hence somewhat problematic. All the programs have been translated into a variety of languages, and a decision must be made as to the extent of translation that is appropriate: should conventions as well as mathematical terms and notation be changed? (Mackrell, 2011a)

The other area in which design decisions are necessary is labeling: geometric objects have been deliberately labeled in Cabri, but were labeled by default in the other programs. By requiring the active creation of labels, the Cabris avoid clutter: the amount of information displayed by GeoGebra may be confusing. However, automatic labeling may give information that the inexperienced user may not think to provide and which may later be important in interpreting and referring to the figure.

Apart from Cabri, measurements are labelled automatically as the measurement is made, and may be later edited. This has the advantage that the labels can give the measurement an immediate meaning. However, conventions for the segment length (\(AB\), \(|AB|\), or \(\overline{AB}\)) must be understood. For the area of the circle (and in general) GSP consistently names measurements
clearly, giving the object measured and also the type of measurement. GeoGebra does not name the circle, and Cinderella’s notation $|C_0|$ for the area of the circle is problematic in English-speaking mathematics.

A final comment to do with creating the model is that in all other programs, replicating the construction of an object such as a segment does not result in a new object being created: in GeoGebra any number of segments can be created between the same points and each is listed as a separate object in the algebra window and when the cursor passes over the segment on the page. This is confusing and problematic in that a fundamental axiom of Euclidean geometry is that there is only one straight line that passes through two points. In the other programs, objects are only considered to be distinct if they have distinct defining objects, although they may coincide if their defining objects temporarily coincide.

**Task 3: Change the radius of the circle, observe the effect on its area and make a conjecture about the relationship between the two**

Unless the axes are hidden, GeoGebra shows the coordinates of the radius point as it is dragged: the other programs enable a focus on the way in which the area changes as the radius is changed without distraction. The measurements move with the figure in Cabri and Cinderella and can be attached to the figure in GSP. In GeoGebra, the segment length is attached, but not the circle area, making it more difficult to see any connection between the two.

It is clear that as the radius increases, so does the area: a conjecture is hence that the area is some multiple of the radius.

**Task 4: Test the conjecture by calculating area/radius**

This method of testing the conjecture will be used to illustrate a powerful connection between calculation and algebraic understanding: the arithmetic operation of dividing one number by another may be generalized to the algebraic operation of dividing one variable by another.

The ability to act as a mediator between numerical and algebraic operations depends upon a number of software design features. The first of these is whether the tool that performs calculations is specifically seen to be a calculator tool. This is the case in the Cabris and GSP, but not the case with Cinderella, which uses a function-defining tool, and GeoGebra, which uses the input bar. These two programs provide an empty entry box: all operations must be entered by typing, and the user is required to know symbols for multiplying, “*”, dividing, “/”, and exponentiation, “^”, which differ from those used on paper or with a calculator.

The most important consideration, however, is the way in which numbers are entered. In GSP, Cabri, and Cinderella, clicking on the number on the screen enters it in the calculator. The user
does not need to recognize that the number represents a variable. In GSP in particular, the name of the measurement is entered in the calculator when the number is clicked and the detail of the calculation with measurement names displayed on the geometry page when the calculation is complete, thus mediating between the idea of dividing two specific numbers and dividing area by radius in general.

In contrast, the GeoGebra input bar only accepts text and hence required that the names of the variables associated with each number be typed in. This demands the awareness first that the measurements are named variables and secondly that one variable might be divided by another, but no means is available to build this awareness. The algebra window, with a large amount of distracting information such as the equation of the circle, needs to be open to find the names of the variables and the text input requirement creates issues with syntax, made more difficult by the confusion described below regarding the segment name, which behaves as a variable in the calculation. It was also necessary to be aware of further syntax in order to display the result on the geometry page, rather than lose it in a new unidentified variable in the algebra window.

Opening the algebra window and showing the construction protocol highlight a fundamental mathematical error in GeoGebra. The letter \(a\) is treated as a variable and assigned a value of 1.44 in the algebra window. However, it is clear from the construction protocol that \(a\) is the name of a geometric object, i.e. the segment joining points A and B.

![Figure 2 Geogebra Algebra Window and Construction Protocol](image)

Using a name as a variable is a common error in the early stages of algebraic understanding (Küchemann, 1981), but also one that may persist into university and adversely affect performance in undergraduate mathematics courses (Gray, Loud & Sokolowski, 2005). This error is hence likely to create difficulties for students.

This is a deliberate design decision, however: “the algebraic representation of a segment is its length” (Hohenwarter, personal communication, 9th September, 2010). Another issue is that the length is simply one of many measurements that could be made of the segment and is not sufficient to define it. The actual algebraic representation of a segment is of the form
\{(x, y) \mid y = mx + c, p \leq x \leq q\}$, where the line defined by the endpoints of the segment has equation $y = mx + c$ and $p$ and $q$ are the $x$ coordinates of the endpoints of the segment. Similar problems are that the name of the polygon is equated to its area, and points are equated to their coordinates, which simply give the temporary location of a point rather than its defining characteristics (for example the point may be defined as the intersection of two lines).

The other programs do not make such mistakes: Cinderella does not attempt to give algebraic definitions of objects that do not have a clear equation. In Cabri and GSP, the equation or coordinates of appropriate objects relative to particular axes can be found using a measurement tool and are not considered intrinsic to the object.

Further information concerning the figure descriptions of the programs is given in Mackrell (2010a): sailent points here are that the GeoGebra algebra window is the only one to list objects alphabetically, useful for locating an object with a known name, but problematic in not giving objects in the order of construction. A newly created object, assigned a name automatically, may be particularly difficult to locate, particularly as the construction protocol in which objects are listed in order of construction, does not allow the highlighting of corresponding objects in the geometry window (possible in all but GSP).

The figure descriptions also highlight a further design decision concerning the modes of possible interaction with objects. Unlike Cabri and GSP, Cinderella and GeoGebra distinguish names, which appear in the figure descriptions, from labels, which appear on the page. For example, the area of the circle in Cinderella has the name “A0” but the label “|C0|”, and in GeoGebra has the name “areac” and the label “Area”. This is because the names of objects may be used as input, and hence an object has a name for selection and a label or caption, which may or may not be identical to the name, for identification and description. In GeoGebra, the names of objects must be used in the input box, and objects may have an additional label and caption. In the Cabris and GSP, objects are selected only by clicking on the screen and hence any labels given serve only to identify and describe objects. The conceptual difference between a name and a label may not be easy to grasp.

In contrast, GSP and the GeoGebra algebra window are the only figure descriptions to make the useful distinction between dependent and independent objects, and the GeoGebra algebra window is the only figure descriptions to enable objects to be selected to either be modified or used in future construction¹.

¹ Cabri 3D, not otherwise described here, also has this capability
Having performed the calculation, it is clear that the result changes as the radius changes: the conjecture concerning a linear relationship was incorrect.

**Task 5: Make a table of data and plot the data**

All of the programs allow tables to be made in which measurement values may be recorded. In Cabri II Plus and GSP, creating the table is simple: choose the table tool and click to select the measurements or calculations to be tabulated. If measurements have names, then these names are used as column headings; if not, headings are not given. Geogebra again requires a specific awareness of the measurements as algebraically represented variables and has a more complicated and rather unintuitive data collection technique. Using text input, a point must first be defined (and is hence plotted) with coordinates given by the variables for radius and area. The coordinates of the point may then be recorded in a spreadsheet. This is problematic if data values are not paired. An advantage, however, is that further calculations can be performed on the data within the spreadsheet view.

GSP allows the data from the table to be easily graphed within the geometry page. Axes are constructed with an appropriate scale and all data is displayed. Cabri II Plus allows the data to be copied and pasted into a spreadsheet, from whence it can be graphed and further analyzed, but it does not allow this to take place on the page itself. Geogebra is again unintuitive: the data must be turned into a list, which automatically plots it, but the scale is not adjusted to ensure that all data is displayed.

In Cinderella, the inbuilt programming language Cindyscript must be used to both gather data and represent such data graphically.

**Task 6: Use the measurements to create a graph of area against radius**

In this task, a new representation of the mathematical situation was created to give further insight on the relationship between the two relevant variables. One of the most powerful features of IGS programs is the ability to visually represent the way in which measurements are related: a graph of area against radius could be constructed directly from the existing measurements without needing to either collect and plot data or to define an algebraic relationship.

It is possible in all programs to show inbuilt coordinate axes and then directly plot the point representing (radius, area). GSP is the simplest; the point can be plotted directly by selecting the appropriate tool and clicking on the values on the page. Cabri II Plus requires a geometric construction, for which a custom tool may be created. In GeoGebra, the required point was already plotted (through text input) in order to collect data. Cinderella requires knowledge of a specific syntax to set the coordinates of a pre-existing point.
Once the point was plotted, each program enabled it to be traced, to create a visual record of the way in which area varied as radius was changed. The set of all possible points representing (radius, area) could then be obtained by creating a locus, which represented the graph of area against radius.

In order to create this locus, it was necessary to redefine the radius point as a point on a path. A segment was constructed from the centre A to some arbitrary point C and the radius point B was redefined to be a point on this segment. This was straightforward with Cabri, Cinderella and GSP, requiring the use of a single tool and selection of the point and the segment by clicking. Redefining the point in GeoGebra was more difficult, requiring accessing the properties of the object and typing in a new definition with a specific syntax.

A problem that then arises, however, is that the area increases much more rapidly than the radius, so that the graph is only visible for small values of the radius and it is difficult to make any conjectures concerning its shape. This can be seen in figure 3a below. In Cabri and GSP, this can be corrected by changing the scale on the y axis: the circle remains the same, but the graph looks more like that in figure 3b. In Cinderella it was only possible to change the scale by zooming in or out. Geogebra allows rescaling, but with the consequence that the circle changes shape, as shown in figure 3b below.

**Figure 3a** GeoGebra circle and graph with equal axis scaling

**Figure 3b** GeoGebra circle and graph with unequal axis scaling

This is a consequence of the design decisions to use a fundamentally algebraic representation of objects and to allow axes to be scaled independently. In this representation, the plane represented by the screen is considered to be based on a coordinate system, to which all objects will automatically be referred. The circle is defined by using the coordinate locations of its center and radius point to generate its equation. This equation is considered to be fundamental; if the
axes change, then the equation is preserved but the appearance or location of the circle will change. This is the position taken by GeoGebra and Cinderella.

An alternative possibility is to create a visual simulation of a Euclidean plane on the screen. On this plane, distance is defined by a rigid ruler and location is not a fundamental property of objects. A coordinate system may be introduced but is not intrinsically linked to geometrically created objects: the system is simply another object on the plane, which may be used to give geometrically defined objects a relative location and algebraic representation when appropriate. It can also be used to define objects algebraically. The behaviour of an object is dependent on the way in which it is defined: as axes change a geometrically defined circle does not change in visual location or appearance, although its algebraic representation relative to the axes will change, but an algebraically defined object will change in visual location or appearance, but not in algebraic representation. This is the position taken by Cabri and GSP.

Both fundamental representations give equivalent visual models of a Euclidean plane – provided that the axes used to define objects always have the same relative scale, which is the case in Cinderella. In contrast, in GeoGebra the axes can easily be independently scaled, with the consequence that the visual model breaks down. Instead, distances appear to depend on direction and angles appear to depend on orientation, which more advanced learners may find interesting to explore, but which may be highly problematic for learners still dependent on visual representation such as 12 or 13 year olds learning about the area of a circle. It is also problematic when modeling real-world objects or for example using the unit circle to define trigonometric ratios.

**Task 7: Find the equation for this curve**

Using the Cabri coordinates and equation tool, it was possible to find the equation of the locus directly. This was not possible with any of the other programs: GeoGebra did not even list the locus as an object in the algebra window.

All programs will fit to the locus a graph defined by means of an algebraic equation. Cabri II Plus requires an expression to be defined and applied to an axis, whereas GSP, GeoGebra and Cinderella require the definition and plotting of a function. The shape of the graph suggests that it might be fitted by a quadratic function and hence the graph of, for example, \( y=3x^2 \) could be plotted. The number 3 could be modified either by changing the definition of the function or expression, or by introducing a parameter \( p \) to replace the 3. GeoGebra has a particularly simple means of introducing a parameter by means of a slider object: sliders in the other programs must
be constructed geometrically. Manipulating this parameter will give the curve of best fit as approximately $y = 3.1 \times x^2$.

**Task 8: Test the formula found**

One way to test this formula would be to edit the calculation of area/radius to area/radius^2, which might suggest that 3.1 is only accurate to one decimal place. It was also possible to create a function or expression $3.1 \times r^2$, substitute the radius for $r$ and compare the result with the measured area. Cabri used the simpler language of evaluating an expression, and the other programs used the language of functions, with text input needed for GeoGebra.

**Task 9: Justify the formula found**

This task, crucial pedagogically, is beyond the scope of this paper. However, files can be constructed which, while not proving conclusively the formula found for the area of a circle, may help students to understand why this formula works. Screenshots from one such GeoGebra file are shown below (Dording, 2009). Such a file could also be constructed in the other IGS programs.

**Figure 4** Visual demonstration of the connection between the radius and area of a circle

**Conclusion**

All of the programs had the necessary affordances to perform the tasks above, but there are large variations in the way in which the affordances have been implemented. Cabri did not label objects automatically and needed to use a spreadsheet to plot and analyze data, and a custom tool to plot a point given its coordinates, but uniquely had a tool to find the equation of a locus. Both Cabri and GSP were simple to use, with GSP identifying measurements and using measurement names in calculations in a way that would mediate the idea of calculations involving variables. Cinderella had a number of issues, including conventions for measurement names, rounding, some syntax and the inability to scale axes. On the other hand, Cinderella has an exceptionally powerful inbuilt programming language.
GeoGebra had some useful features, such as the distinction between dependent and independent objects in the algebra window, the ability to select objects for geometric construction by clicking on their names in the algebra window, the ability to perform calculations with table data and the ability to easily define a slider. However, there were also a large number of problems. A major issue is the prevalence of text input, which requires the use of named variables, is open to syntax errors, and requires the algebra window to be open in order to locate variables, displaying a large amount of information irrelevant to the task. A serious mathematical mistake is made in using a geometric label as a variable. A second mistake is that an arbitrary number of identical objects can be created from the same initial objects. The inability to define an object without reference to coordinate axes means that the shape of the object is dependent on the relative scales of the axes. Minor issues include cluttered labeling, potential confusion between names and labels, the label for the circle area not referenced or attached to the circle, a lack of units in measurement, problematic rounding, calculation not requiring a calculation tool, coordinates of dragged points being displayed if axes are visible, difficulty in placing the result of a calculation in the geometry window, the inability of the algebra window to show objects in order of construction, needing to plot a point to collect data, data only being collected in pairs, unintuitive data plotting, complicated redefinition, and loci not being listed in the algebra window.

Unlike Cinderella, GeoGebra claims to be “from elementary school to university level”. This, and its increasing popularity among teachers, make the issues noted here of some concern. Greater attention needs to be paid to design decisions and issues.

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TOWARD A COLLECTION OF GEOGEBRA CALCULUS APPLETS

Marc Renault
Shippensburg University, US

Note: Paper on Innovative Practices and Development Activities

Abstract

During the fall of 2010, I had the opportunity to devote a sizeable amount of time creating a collection of online applets for use in a typical college Calculus I class. In this paper I describe my experiences using GeoGebra to construct these applets. I describe the need for such a collection, my design considerations, some of the obstacles I encountered, and my own assessment of the project. Furthermore, I class-tested these applets in spring 2011, and I report on the success of their class use. Before reading any further, the reader is strongly encouraged to explore these applets. The collection is available at http://webspace.ship.edu/msrenault/GeoGebraCalculus/GeoGebraCalculusApplets.html

Keywords: Calculus, applets, learning through exploration, student activities, in-class demonstrations, JavaScript

Need

Is there really a need for more Calculus applets on the Web? Surely by now the Internet must be practically overflowing with quality Calculus demonstrations and activities, right? This is what I thought at the beginning of 2010, but Googling “Calculus Applets” quickly disabused me of that notion. There are many Calculus applets available, but (1) there are few sizeable collections suitable for an entire Calculus course, (2) many applets use older Java technology which looks dated and clunky, and (3) many applets are just not well made, following a “more features make a better applet” approach, and finally (4) many applets do not have solid pedagogical accompanying text.

To be fair, the collection at http://www.calculusapplets.com by Thomas S. Downey is quite good, and its existence gave me reason to think that there was no need for me to create my own collection. However, I ultimately decided to proceed because GeoGebra offers an aesthetic and usability that I find more appealing.

Goals of the Project

I wanted a collection of Calculus applets that solved some of the shortcomings I observed. I desired the following characteristics:

• the collection should be organized and complete, closely following standard curricula
- applets must have aesthetic appeal
- each applet should have a defined, focused goal, and not get lost in “feature overload”
- applets should have appropriate accompanying text and explorations
- applets should be easy to use – in fact, one should intuit their use without having to read through explanations
- applets should be suitable for student exploration outside of class
- applets should be suitable for instructor in-class demonstrations.
- the applets should run in any web browser without requiring any extra software
- the applets should be easy for me to create and modify

I decided to use GeoGebra because it creates aesthetically-pleasing applets and it is easy to use. Moreover, the fact that it is free software means that students who are interested can continue to explore it further on their own.

**Results of the Project**

By the end of fall 2010, I had created 44 applets together with explanatory text and accompanying exploration activities. Most are in the Calculus topic of differentiation; the integration treatment could still benefit from further development.

The design of each applet page follows an applet-explanation-exploration format; I thought it was important to put the applet at the top of the page to grab the user’s attention right away. This also facilitates its use for in-class demonstrations.

While the applets are written using GeoGebra, I often incorporated JavaScript in the HTML page to increase the functionality of the applet. There are many times that nice effects were achieved by using JavaScript to make several applets “communicate” with each other on the same page.

**Difficulties**

There were several unexpected difficulties I encountered in this project. The first is the fact that it is not easy at all to create mathematical text in an HTML page. In order to incorporate applets and JavaScript into the HTML page, I decided that I would edit the HTML code directly instead of using a WYSIWYG editor – this compounded the difficulty of incorporating mathematical text. In fact, the first few weeks of my project were dedicated to finding a way to make this work well. After considering (and having difficulties with) JSMath, ASCIIMath, and MathJax, I ultimately ended up writing a JavaScript file that converted TeX-like HTML code into formatted HTML code using CSS. It may not be the most elegant solution, but it gave me good control over how the mathematical notation appeared.
A second difficulty is that buttons, script, text-boxes, and drop-down menus are not built into GeoGebra itself (yet). I would have preferred putting buttons on the applet stage itself, but I settled for putting them just next to the applet instead, and using JavaScript to handle the button clicks.

Although not technically a “difficulty” I had a sudden dearth of inspiration when I started trying to think up applets to accompany more computational topics such as the product rule and chain rule. GeoGebra’s graphical strengths did not apply here. One could make graphical applets to accompany the rules, but I doubt that such graphical demonstrations would add significantly to a student’s understanding or ability of the rules.

**Classroom testing**

I incorporated these applets into my teaching of Calculus I during spring semester 2011.

**Final Reflections**

I am very pleased with where the project currently stands, but I am also open to suggestions for further improvement. Perhaps this is one of the greatest strengths of the Web: pages may be improved at any time. GeoGebra was well-suited to the task, and the results were beautiful applets and beautiful mathematics.

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“Mathematics is … not physical, not mental, but social”. It is “a social-cultural-historical entity” (Hersh, 1997, pp.248-9). Although mathematical progress does at times involve periods of intense individual work, an examination of the discipline’s history shows that growth has come from each contributor building on the work of their predecessors and contemporaries. This is illustrated most vividly by the 350 years of efforts to construct a proof to Fermat’s Last Theorem (Singh, 1997), during which multiple mathematicians combined ideas from previous generations to progress towards a solution. The larger social nature of this puzzle can be seen in the fact that news of Wiles’ solution appeared on the front page of the June 24, 1993 edition of The New York Times (Kolata, 1993) and was featured in many other newspapers around the world. Mathematics is a shared human enterprise and today for many, particularly students in schools and universities, information sharing means interaction via the Internet.

GeoGebra, written in Java, has natural links to the Internet or web. In fact those without the software installed on their computers can run the program as an application within a web-browser window by going to the URL http://www.geogebra.org/webstart/geogebra.html. In addition, GeoGebra, via the menu sequence File > Export > Dynamic Worksheet as Webpage (html)…, allows users to generate html files that hold Java applets for modified versions of GeoGebra. Teachers, by reducing the range of user options and providing tools constructed within GeoGebra, have used this option to produce interactive student learning activities. Two examples of such resources can be found at http://www.uff.br/cdme/jct/jct-html/jct-en.html and http://queensgeogebra.pbworks.com/f/Score_a_Basket.html.

But, the GeoGebra instruction “Dynamic Worksheet as Webpage” may be somewhat of a misnomer or possibly lead us off along a restricted path. Yes, this function can be used by instructors to generate effective online worksheets for students, but possibly more importantly any GeoGebra user can employ the tool to produce a webpage that displays the results of their work. Once this file is mounted on a website the GeoGebra work is available for viewing and additional manipulation by others interested in the original mathematical problem or exploration. At Madeira
High School in Cincinnati, Ohio, students in the Honours Geometry course share their solutions to problems with classmates and all web users by posting GeoGebra applets to a wiki ([http://madeiramath2.wikispaces.com/](http://madeiramath2.wikispaces.com/)) established and maintained by their teacher Steve Phelps. Such online GeoGebra sharing by students can be expanded into full collaboration by mechanisms that support responding to and building upon previously posted work.

Research in GeoGebra supported online asynchronous collaboration has been studied by Stahl (Stahl, Ou, Cakir, Weimar & Goggins, 2010) in the Virtual Math Teams project associated with the Problems of the Week resource ([http://mathforum.org/problems_puzzles_landing.html](http://mathforum.org/problems_puzzles_landing.html)) at the Math Forum. In this environment students interact via text chat while working on a problem. In this they are supported by a shared whiteboard for displaying their work, and a multi-user version of GeoGebra. Manipulations on the whiteboard and within GeoGebra are seen by all group members online and can be graphically linked to comments within the chat. Online collaborative work using the Virtual Math Teams tools has been found to “be an exciting, engaging, motivating and rewarding experience” (p. 130) for students as they solve problems that can be addressed over a limited timeframe. For larger sustained investigations online asynchronous tools may be required.

**Mathematical Collaboration with Wikis and GeoGebra**

Roulet (2010) has experimented with combining a wiki structure (MediaWiki, [www.mediawiki.org](http://www.mediawiki.org)) with GeoGebra to create an environment in which students can work in asynchronous mode to collaboratively explore mathematics problems. The planned combination of MediaWiki and GeoGebra would provide a space where a teacher could initiate a group problem solving activity by creating a new wiki page describing a problem and embedding a GeoGebra applet as a starting point. Students could then contribute to a solution by adding linked pages that contain a revised GeoGebra window and in text provide commentary on the changes they made. A student, while studying the contributions of his or her collaborators, could at any point grab a GeoGebra instance that they viewed as productive, create a new linked wiki page, paste in the GeoGebra file and then use this as a starting point for further investigation, along with providing a text message describing their actions and thinking. Since this contribution could be inserted at any point in the conversation there would be the potential for branching and multiple investigation paths (see Figure 1).
Efforts in this project have so far led to linking of GeoGebra applets to MediaWiki pages, but the production of the Java code and the links has not been automated. Users must export their GeoGebra work as a Dynamic Worksheet, upload the html file, and then add the required linking. Since both MediaWiki and GeoGebra are open source software providing access to the underlying code it should be possible to add functionality automating these processes, but this is proving to be a more complex task than originally imagined.

While working on the MediaWiki based resource we have also implemented a prototype of the planned environment using a PBworks (http://pbworks.com/) wiki, a tool familiar to many teachers and students. As in MediaWiki a collaborative exploration results in a network of linked pages each setting out solution suggestions and holding an illustrating GeoGebra sample (see Figure 1). To explore an online version of the network displayed in Figure 1 go to:

Here in PBworks users must again locally save a Dynamic Worksheet containing their GeoGebra applet, upload this to the wiki, and insert an appropriate link from a page containing their explanation. In PBworks this is all relatively straightforward and likely something familiar to many high school and university students. Although the individual steps should not be a problem it remains to be seen if the time they consume is an impediment to student use and free flowing collaboration.

**Adding Video for Student Explanations**

Increasingly sharing across the web is taking place via video. At the end of 2010 Canadian high school and college/university age Internet users were on average spending close to 20 h per month while viewing in excess of 200 online videos (comScore, 2011). Thus students might find it more “natural” to present their thinking and GeoGebra manipulations in video format rather than text on a wiki page. Jing (TechSmith, 2010) and the related website, [www.screencast.com](http://www.screencast.com) makes this easy to accomplish and at no cost. With Jing installed on their computer a student can record a video of a GeoGebra window as they work on a problem solution and in audio provide the thinking behind their actions. The resulting video, stored in the student’s private folder at [www.screencast.com](http://www.screencast.com) can be embedded into a wiki page along with the link to an applet holding GeoGebra in the state it was left at the end of the video. Go to: [http://collabmath.pbworks.com/w/page/40790294/model-of-simple-online-collaboration-with-video](http://collabmath.pbworks.com/w/page/40790294/model-of-simple-online-collaboration-with-video) to view the wiki pages of a problem solving activity parallel to that shown in Figure 1, but with the students presenting their thinking within Jing videos.

**Beyond the Technology**

In 2004 Nason and Woodruff, in discussing the problems faced in computer supported collaborative learning in mathematics, argued that most school mathematics problems do not engage students in extended conversations and online collaborative learning environments do not support the simple exchange of mathematical ideas in symbol or image forms. The experiments to date, related above, show that the technical problem identified by Nason and Woodruff has essentially been addressed. Tools exist to support online GeoGebra collaborative explorations and problem solving. Combining wikis, GeoGebra, and video is relatively easy. But a technical solution is not sufficient.

Research (Roulet 2011) examining student collaboration within Math-Towers ([www.math-towers.ca](http://www.math-towers.ca)), a website designed for students in Grades 6 to 10, shows that providing tools for mathematical investigations and user sharing does not guarantee successful collaboration. Students need to arrive at the website with prior classroom experience with open-ended problem
solving and the view that progress comes from all participants contributing suggestions that may
not be fully formed or obvious routes to a solution. Teacher coaching to encourage the posting of
initial ideas is often required. In addition, as noted by Nason and Woodruff (2004) there is a need
for problems that are accessible by the student and at the same time sufficiently complex and
interesting to sustain student motivation over a period of several days. Thus those wishing to
develop online environments to support asynchronous collaborative mathematics exploration
need to address technical, social, pedagogical, and curricular issues.

Questions to be Explored

Below is a non-exhaustive list of issues and question that might be explored by a working group
wishing to provide students with a GeoGebra based online environment for collaborative
mathematics investigations.

A. Technical

1. Medium of communication for student explanations – text, voice, video
2. Presently available tools that could be combined – GeoGebra, wikis, video and audio
   support
3. Level of automation required – Can we expect students to handle saving, uploading, and
   linking?

B. Social

1. Group composition and size
2. Social norms required – commitment to the group, respect for the contributions of others

C. Pedagogical

1. Prior student experience required – technical and problem solving
2. Teacher support and intervention possibly required
3. Balance between teacher coaching and student freedom to explore

D. Curricular

1. Appropriate grade levels and topics
2. Problems and investigation roots that can sustain student interest

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CLASSICAL GEOMETRY WITH GEOGEBRA

Petra Surynková
Charles University in Prague
Faculty of Mathematics and Physics, Czech Republic
petra.surynkova@seznam.cz

Abstract
This contribution addresses the application of dynamic system GeoGebra in teaching and learning geometry. Our aim is to increase the interest of students in studying classical geometry at secondary schools and colleges. One possible approach of improvement in studying geometry is the integration of computer software in the teaching process. This way seems to be interesting, attractive and motivational for students. Indeed the usage of computers in education is very modern. We will use GeoGebra for visualization, for the proving geometric problems in the plane or for the demonstration practical uses of geometry. The usage of software will be shown on some concrete examples. We will demonstrate the advantages of dynamic geometry system on examples from the field of kinematic geometry. We will compare GeoGebra with other systems for geometric modeling with respect to accessible functions. We will mention how to use geometric systems for image creation or illustrations of geometric problems. The outputs can be used in publications and also for home schooling and e-learning.

Key words: kinematic geometry, classical geometry, motion

1 Introduction and motivation
Geometry, the study of properties and relations of geometric figures, is an important and essential branch of mathematics. Geometry is important for everyone, not only for technicians, designers, architects, builders or civil engineers. We all need good visual imagination in our everyday life as well. The two and three dimensional shapes which surround us are originated in geometry. The world we live in is influenced by geometry. If we know how to understand and apply the relationship between shapes and sizes we can use it more efficiently. Some people think in images and shapes so they need the understanding of geometry to be able to do that.

Without the use of geometry the great works of artists, painters and builders would only have stayed in ideas and dreams.

According to (Hilbert, 1999) the study of geometry develops logical reasoning and deductive thinking which helps us expand both mentally and mathematically. If we learn to use geometry we also learn to think logically. It’s very important in everyday life – many difficult problems can be erased and the simple solutions can be found. Students can often solve problems from other fields more easily if they represent the problems geometrically.
Geometry is useful for learning other branches of mathematics and it can also be used in a wide range of scientific and technical disciplines. Some scientific branches require direct knowledge of geometry.

Geometry can be conceived as an independent discipline with many branches – Euclidean geometry, differential geometry, algebraic geometry, topology, non-Euclidean geometry and so on. The main field of our interest is the study of classical geometry and descriptive geometry – geometric constructions, projections, geometry of curves and surfaces. Mainly it is geometry which allows the representation of three dimensional objects in two dimensions. Classical geometry is geometry of the Euclidean plane and space.

The remainder of this paper is organized as follows: the section 2 is devoted to current problems of the unpopularity and the difficulty of studying geometry. The main subject of the section 3 is how to increase the interest of students in studying classical geometry. In the section 4 we will describe kinematic geometry and in the section 5 we will introduce some definitions and examples of special motions. We will also demonstrate the advantages of the dynamic geometry system GeoGebra on concrete examples from the field of kinematic geometry.

## 2 The study of geometry

The study of geometry can be very difficult. This branch of mathematics isn’t popular among students. Drawings (the results of geometric projections) are sometimes very difficult to understand. For that reason geometric problems must be provided with clear examples. Intuitive understanding plays a major role in geometry. With the aid of visual imagination we can illuminate the problems of geometry. It is possible in many cases to show the geometric outline of the methods of investigation and proof without entering needlessly into details. The problem can be more understandable without strict definitions and actual calculations. Such intuition has a great value not only for research workers, but also for anyone who wishes to study and appreciate the results of research in geometry. Of course if we understand the main principles of a problem then we can use exact definitions.

The currently predominant view among students and the general public is that classical geometry is not important and useful. Drawings of classical geometry can be replaced by the outputs of modern computer software. Of course, computers can help us solve geometric problems and increase the efficiency of our work but we still have to know the basic principles and rules in geometry.

Is it possible to learn geometry? Yes, but it would be easier for students if they had encountered classical geometry, constructions and geometric proofs earlier. Sometimes students of technical
specializations experience geometry only at college. That is too late. We work mainly with undergraduate students, so what can be done to make college geometry more comprehensible? How to increase the interest of students in studying classical geometry at secondary schools and colleges? This is the main subject of this article.

3 How to increase the interest in studying geometry; use of computers in the teaching process

Our aim is to increase the interest of students in studying classical geometry at secondary schools and colleges. One possible approach of improvement in studying geometry is the integration of computer software in the teaching process. This way seems to be interesting, attractive and motivational for students. Indeed the usage of computers in education is very current. Computers influence our everyday life including geometry. We have to follow the general trend.

Nowadays, computer-aided design is commonly used in the process of design, design documentation, construction and manufacturing processes. There exist a wide range of software and environments which provide the user input tools for modeling, drawing, documentation and design process. These software and environments can be used to design curves and geometric objects in the plane and curves, surfaces and solids in the space. According to the applications more than just shapes can be involved. In modern modeling software we can also work with rotations and other transformations; we can change the view of a designed object. Some software provides dynamic modeling. Technical and engineering drawings must contain material information and the methods of construction. Computer-aided design is used in numerous fields: industry, engineering, science and many others. The particular use varies according to the profession of the user and the type of software.

These modern methods which are widespread in various branches can be useful in the teaching process, too. We can prepare students for their future employment. We still place emphasis on the understanding of the principles used in geometry.

I have my own experience of teaching classical geometry, descriptive geometry and computational geometry at universities – Charles University in Prague – Faculty of Mathematics and Physics and Czech Technical University in Prague – Faculty of Architecture. College mathematics and geometry is very difficult for many students. It is necessary to motivate and to arouse their interest in geometry. As was mentioned above, it is necessary to improve the teaching of geometry at elementary schools and at secondary schools.

In my lessons I use computer software for visualization, for the proving of geometric problems in the plane and in the space or for the demonstration of the application of geometry in practice. I work for example with Rhinoceros - NURBS modeling for Windows (Rhino), Cabri II Plus, Cabri 3D and of
course with GeoGebra. Use by teachers and students is always free of charge, it is the great advantage of GeoGebra. Consequently it can be used by students for home schooling and e-learning. Nowadays we have extensive database of geometric tasks, images and 3D models – the outputs of these software.

I use GeoGebra for creation of stepwise guides through geometric construction which can help my students understand the problem in intuitive and natural way. Moreover I show special constructions applied in descriptive geometry and due to included functions and tools students can discover proofs more easily. In this contribution we will demonstrate the advantages of dynamic geometry system on examples from the field of kinematic geometry. Good geometric imagination and perception is very important for understanding constructions in geometry. It is not possible to learn the constructions by heart, we have to understand geometric problems. Let us start with the theoretical part of kinematic geometry. It is necessary for further understanding because we will explain important terms. Then we will introduce definitions and examples of curves which are created by special motion of geometric objects.

4  **Kinematic geometry**

Kinematic geometry in the plane is a branch of geometry which deals with the geometric properties of objects which are created by motion of moving plane. Geometric motions – that is without regard to the cause of the motion, velocity and acceleration. Theoretical kinematics is a large subject and it is not possible to treat it completely in this article. We will restrict to essential basics.

We consider unbounded infinite plane which contains geometric elements (points and curves). We mainly treat such elements as points, straight lines, circles and line segments and study the geometric properties which arise as the move in the plane. Especially we are dealing with aspects of transformation geometry. We consider those transformations in the Euclidean plane such that all distances remain fixed during the motion.

The treatment in this article is geometrical thus only geometric interpretations are given.

4.1  **The determinations of geometric motion**

Let $\Sigma$ be the moving plane which slide over the fixed plane $\Pi$. The moving plane $\Sigma$ contains curves and points which are at each instant of the motion identical. In the fixed plane there are generated the roulettes and the envelopes. Any point of the moving plane $\Sigma$ describes a curve, its path (often called roulette), in the fixed plane $\Pi$. Any curve of the moving plane $\Sigma$ describes a curve, its envelope, in the fixed plane $\Pi$. The path is the locus of a point in the moving plane. Geometrical envelope of a family of curves in the moving plane is a curve which at each of its
points is tangent to a curve of the family. Let $\Sigma^1, \Sigma^2, \Sigma^3, \ldots$ denote the sequence of positions of the moving plane $\Sigma$. The positions in the moving plane of points $A, B, C, \ldots$ will be $A^1, B^1, C^1, \ldots$ when $\Sigma$ is at $\Sigma^1$ and $A^2, B^2, C^2, \ldots$ when $\Sigma$ is at $\Sigma^2$. It is analogous for curves. Figure 1 shows an example of the motion of the moving plane.

![Figure 1](image)

**Figure 1.** An illustration of the moving plane $\Sigma$ containing points $A, B, C$ (with line segments) which slide over the fixed plane $\Pi$. Points $A, B, C$ describe curves $\tau_A, \tau_B, \tau_C$ - the paths. $\tau_A, \tau_B$ are given, $\tau_C$ is obtained by moving. Indices denote positions of the moving plane. All distances remain fixed.

Let us discuss the determination of the motion in the plane. There are several possibilities how to define the motion:

a) **The motion is completely determined by paths $\tau_A$ and $\tau_B$ of two points $A$ and $B$ (end points of segment line).** These points are in the moving plane, the paths are in the fixed plane. The following equations are satisfied: $|A^1B^1| = |A^2B^2| = |A^3B^3| = \ldots$ See figure 2.

b) **The motion is completely determined by envelopes $(m)$ and $(n)$ of two curves $m$ and $n$.** These curves are in the moving plane, the envelopes are in the fixed plane. The following equations are satisfied: $|\varepsilon m^1 n^1| = |\varepsilon m^2 n^2| = |\varepsilon m^3 n^3| = \ldots$ See figure 3.

c) **The motion is completely determined by envelope $(m)$ of curve $m$ and path $\tau_A$ of point $A$.** The following equations are satisfied: $|A^1m^1| = |A^2m^2| = |A^3m^3| = \ldots$ See figure 4.

In special cases the envelope of a family of curves in the moving plane degenerates into point. These situations are shown in figures 5 and 6.

The proofs of these theorems can be found in (Bottema & Roth, 1979) and (Lockwood, 1967).
Figure 7. The motion is given by paths $\tau_A$ and $\tau_B$ of two point $A$ and $B$.

Figure 8. The motion is given by envelopes $(m)$ and $(n)$ of two curves $m$ and $n$. 
Figure 9. The motion is given by envelope \((m)\) of curve \(m\) and path \(\tau_A\) of point \(A\).

Figure 10. The motion is given by envelopes \((m)\) and \((n)\) of two curves \(m\) and \(n\) which degenerate into points.
Figure 11. The motion is given by path $\tau_A$ of point $A$ and envelope $(m)$ of curve $m$ which degenerate into point.

4.2 The centrodes

We shall characterize any motion of the plane only by the initial and final positions. Of course, we can get from the initial to the final position in different ways. It will be one of several tasks to find the simplest possible way of an effecting any given motion.

The simplest motions in the plane are translations in which every point of the plane moves through the same distance in the same direction and every straight line remains parallel to its initial position. Another well-known type of motion is the rotation of the plane through a given angle about any given point. The direction of every straight line is changed by the given angle and the centre of the rotation is the only point of the plane that remains fixed.

It is possible to prove that every motion of the plane can be carried out in one translation or one rotation. This fact considerably simplifies the study on geometric motions in the plane. More detailed information can be found in (Bottema & Roth, 1979).

We can consider translations as rotations through the angle zero about an infinitely distant point. If we adopt this point of view, we may regard any motion of the plane as a rotation through some definite angle which is zero in the case of translation. Several theorems and the possibilities of compositions of simple motions are discussed in details in (Bottema & Roth, 1979).
Now the motion in the plane is given. We assume that $\Sigma^i$ and $\Sigma^{i+1}$ are two positions of the moving plane $\Sigma$. The change of position $\Sigma^i \rightarrow \Sigma^{i+1}$ is associated with a centre of rotation $S^i$. If we consider a limiting position of $\Sigma^i$ and $\Sigma^{i+1}$ (the difference between $\Sigma^i$ and $\Sigma^{i+1}$ becomes smaller and smaller) the point $S^i$ is called the instantaneous centre of the motion which is related to instant $i$.

**Definition 1.** The locus of the instantaneous centres at every moment of the motion is a curve in the fixed plane. This curve is called the fixed centrode of the motion. ■

But in the same motion we may also regard the plane $\Sigma$ which we had considered movable as fixed and the plane $\Pi$ which we had considered fixed as moveable. That is we may interchange the roles of the two planes. This motion is called the inverse motion. The original motion is called the direct motion. One motion determines the other one and the inverse of the inverse motion is the direct motion.

**Definition 2.** The locus of the instantaneous centres at every moment of the inverse motion to a given motion is a curve in the moving plane. This curve is called the moving centrode of the direct motion. ■

A more detailed study shows that the motion is completely determined by the form of the two centrodes. At each instant of the motion the two curves are tangent to one another at the instantaneous centre and there is no slipping. The motion is obtained by rolling (without slipping) the moving centrode in the moving plane on the fixed centrode in the fixed plane. If we interchange the roles of the centrodes we get the inverse motion. From the fact that the centrodes roll on each other without slipping it follows that the arc bounded by any two points on the fixed centrode has the same length as the arc bounded by the corresponding points on the moving centrode. There is one more possibility how to define the motion:

**d) The motion is completely determined by the fixed centrode $p$ and the moving centrode $h$.** The following equations are satisfied: $|S^{i+1}S| = |(S^i)(S^{i+1})|$. See figure 7.
4.3 Geometric constructions of the centrodes

We shall discuss geometric construction of the centrodes. Let us consider the example which is given by paths $\tau_A$ and $\tau_B$ of two points $A$ and $B$. We construct the fixed and the moving centrode for this special determination.

Construction 1. The fixed centrode is the locus of the instantaneous centres at every moment of the motion. The instantaneous centre $S^1$ ($S^2, S^3, \ldots$) is the intersection of the normal line to the path $\tau_A$ at the point $A^1$ ($A^2, A^3, \ldots$) and the normal line to the path $\tau_B$ at the point $B^1$ ($B^2, B^3, \ldots$). See figure 8.

Construction 2. The moving centrode is the locus of the instantaneous centres at every moment of the inverse motion to a given motion. We construct the point $(S^1)$ ($(S^2), (S^3), \ldots$) of the moving centrode using congruence of the triangles $\triangle A^1B^1S^2 \equiv \triangle A^2B^2S^2$ ($\triangle A^1B^1S^3 \equiv \triangle A^3B^3S^3, \ldots$). The moving centrodes is constructed in the position $\Sigma$ of the moving plane $\Sigma$. See figure 9.

Special cases of the centrodes will be listed below.
5 Special motions in the plane

Special motions in the plane will be discussed in this section. For the creation of the outputs and images we use GeoGebra. We have web pages (see http://www.surynkova.info/dokumenty/mff/DGIII/DGIII_ukazky.php) with database of examples of motions in the plane; Surynkova P. (2011). We provide the access to this database to our students, so students have dynamic worksheets at their disposal. In dynamic worksheets some parameters can be changed, see the following figures.
5.1 Cyclical motion

As was mentioned above, a plane motion may be defined by its centrodes. The simplest example is the case where both centrodes are circles or one centrode is circle and the second is straight line. These motions are called cyclical motions.

The motions are classified according to the type and relative positions of the centrodes. We first consider the example where a circle is rolled on a straight line. That is the moving centrode is the circle and the fixed centrode is the straight line. The paths of points which are obtained by rolling the circle on the straight line are called cycloids, curtate cycloids or prolate cycloids, generally cycloid. The motion is called cycloidal. The cycloid is the path of a point on the circumference of the rolling circle, the prolate cycloid is the path of a point outside the rolling circle and the curtate cycloid is the path of a point inside the rolling circle; see figure 10.

Figure 15. Cycloidal motion and examples of cycloids. Cycloid $m_1$ (blue), prolate cycloid $m_2$ (red), curtate cycloid $m_3$ (green).

History and applications of cycloids are very important. More details can be found in (Rutter, 2000) and (Lockwood, 1967).

We can interchange the roles of the two centrodes. That is a straight line is rolled on a circle. The moving centrode is the straight line and the fixed centrode is the circle. The paths of points which are obtained by rolling the straight line on a circle are called the involute of the circle. The motion is called involute. The classification of paths is similar to the cycloidal motion. The paths of points...
which are obtained by rolling the straight line on the circle are called involutes, curtate involutes or prolate involutes; see figure 11.

Figure 16. Involute motion and examples of involutes. Involute $m_1$ (blue), prolate involute $m_2$ (red), curtate involute $m_3$ (green).

We now consider the example where a circle is rolled on a second circle. There are three cases. The paths of points which are obtained by rolling the circle on the outside of the fixed circle are called epicycloids. The motion is called epicycloidal. The epicycloid is the path of a point on the circumference of the rolling circle, the prolate epicycloid is the path of a point outside the rolling circle and the curtate epicycloid is the path of a point inside the rolling circle; see figure 12.

The paths of points which are obtained by rolling the circle in the fixed circle are called hypocycloids. The radii of the two circles cannot be equal. The motion is called hypocycloidal. The hypocycloid is the path of a point on the circumference of the rolling circle, the prolate hypocycloid is the path of a point outside the rolling circle and the curtate hypocycloid is the path of a point inside the rolling circle; see figure 13.
The paths of points which are obtained by rolling the circle on the outside of the fixed circle which is inside the rolling circle are called **pericycloids**. These curves are same as epicycloids. The motion is called **pericycloidal**.

Figure 17. Epicycloidal motion and examples of epicycloids. Epicycloid $m_1$ (blue), prolate epicycloid $m_2$ (red), curtate epicycloid $m_3$ (green).
5.2 Elliptic and cardioid motion

Figure 18. Hypocycloidal motion and examples of hypocycloids. Hypocycloid $m_1$ (blue), prolate hypocycloid $m_2$ (red), curtate hypocycloid $m_3$ (green).

Figure 19. Elliptic motion.
The elliptic motion is given by paths $\tau_A$ and $\tau_B$ of two points $A$ and $B$ (end points of segment line) where $\tau_A$ and $\tau_B$ are straight lines. It can be proved that this motion can be defined by rolling the circle in the fixed circle. Radius of the fixed circle is double radius of the moving circle. That is the elliptic motion is special case of hypocycloidal motion. The paths of points which are obtained by this motion are ellipses, segment line or circle; see figure 14.

The cardioids motion is the inverse motion to the elliptic motion. The paths of points which are obtained by this motion are limaçon of Pascal or circle. In special case limaçon of Pascal is cardioid. It can be proved that this motion can be defined by rolling the circle on the outside of the fixed circle which is inside the rolling circle. That is the cardioid motion is special case of pericycloidal motion; see figure 15.

### 5.3 Conchoid motion

A conchoid is a curve derived from a fixed point $\theta$, another curve, and a length $d$. For every line through $\theta$ that intersects the given curve at $A$ the two points on the line which are of $d$ distance from $A$ are on the conchoid. We can get the branches of this curve by the conchoid motion which is given by path $\tau_A$ of point $A$ and envelope $(b)$ of straight line $b$ which degenerate into point. Envelope $(b)$ is the fixed point $\theta$. See figure 16, there are some examples of various paths.
In some publications other terms for the paths and the special motions can be found. We work with the simplest notations.

There exists a wide range of various special motions in the plane. We discussed the most important and the best known. We can also study the properties of curves; we refer the reader to (Abbena et al., 2006) and (Pottman et al., 2007).

6 Conclusion

We discussed possible approaches how to increase the interest of students in studying classical geometry at secondary schools and colleges. The integration of computer software, mainly GeoGebra, in the teaching process we demonstrated on the examples from the field of kinematic geometry. We also use GeoGebra for creation of stepwise guides through geometric construction which can help students understand the problem in intuitive and natural way. Moreover we show special constructions applied in descriptive geometry and due to included functions and tools students can discover proofs more easily. Of course, students also create some examples and tasks themselves.

We have web pages with database of geometric tasks in the plane and in the space. We provide the access to this database to our students (undergraduate).
In future work we will focus on further methods which can improve the teaching process. We plan to extend our gallery of geometric tasks.

References


