A Review on Metric Symmetries used in Geometry and Physics

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This is a review paper of the essential research on metric (Killing, homothetic and conformal) symmetries of Riemannian, semi-Riemannian and lightlike manifolds. We focus on the main characterization theorems and exhibit the state of art as it now stands. A sketch of the proofs of the most important results is presented together with sufficient references for related results.

1. Introduction

The measurement of distances in a Euclidean space $\mathbf{R}^3$ is represented by the distance element

$ds^2 = dx^2 + dy^2 + dz^2$

with respect to a rectangular coordinate system (x, y, z). Back in 1854, Riemann generalized this idea for n-dimensional spaces and he defined element of length by means of a quadratic differential form $ds^2 = g_{ij}dx^i dx^j$ on a differentiable manifold M, where the coefficients g_{ij} are functions of the coordinates system $(x^1, \ldots, x^n)$, which represent a symmetric tensor field g of type $(0, 2)$. Since then much of the subsequent differential geometry was developed on a real smooth manifold (M, g), called a Riemannian manifold, where g is a positive definite metric tensor field. Marcel Berger's book [1] includes the major developments of Riemannian geometry since 1950, citing the works of differential geometers of that time. On the other hand, we refer standard book of O'Neill [2] on the study of semi-Riemannian geometry where the metric g is indefinite and, in particular, Beem-Ehrlich [3] on the global Lorentzian geometry used in relativity. In general, an inner product g on a real vector space $\mathbf{V}$ is of type (r, ℓ, m) where $r = dim\{u \in \mathbf{V} | g(u, v) = 0 \; \forall v \in \mathbf{V}\}$, $\ell = sup\{dimW | W \subset \mathbf{V}$ with $g(w, w) < 0 \; \forall$ non-zero $w \in W \}$ and $m = sup\{dimW | W \subset \mathbf{V}$ with $g(w, w) > 0 \; \forall$ non-zero $w \in W \}$. A metric g on a manifold M is a symmetric $(0, 2)$ tensor field on M of the type (r, ℓ, m) on its tangent bundle space TM. Kupeli [4] called a manifold (M, g) of this type a singular semi-Riemannian manifold if M admits a Koszul derivative, that is, g is Lie parallel along the degenerate vector fields on M. Based on this, Kupeli studied the intrinsic geometry of such degenerate manifolds. On the other hand, a degenerate submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ may not be studied intrinsically since due to the induced degenerate tensor field g on M one can not use, in general, the geometry of $\bar{M}$. To overcome this difficulty, Kupeli used the quotient space $TM^* = TM/Rad(TM)$ and the canonical projection $P : TM \to TM^*$ for the study of intrinsic geometry of M. Here $Rad(TM)$ denotes the radical distribution of M.

In 1991, Bejancu-Duggal [5] introduced a general geometric technique to study the extrinsic geometry of degenerate submanifolds, popularly known as *lightlike submanifolds* of a semi-Riemannian manifold. They used the decomposition

$TM = Rad(TM) \oplus_{orth} S(TM)$,

where $S(TM)$ is a non-degenerate complementary screen distribution to $Rad(TM)$ and $\oplus_{orth}$ is a symbol for orthogonal direct sum. $S(TM)$ is not unique, however, it is canonically isomorphic to the quotient bundle $TM^* = TM/Rad(TM)$.

There are three types of metrics, namely, Riemannian, semi-Riemannian and degenerate (lightlike). The properties of Riemannian metrics which come from their non-degenerate character remain same in the Semi-Riemannian case. However, neither " *geodesic completeness*" nor "*sectional curvature*" nor "*analysis on Lorentzian manifolds*" works in the same way as in the Riemannian case. However, the case of degenerate metric is different (see Section 5).

One of the widely used technique is to assume the existence of a metric tensor g with a symmetry as follows: Consider (M, g, V) with the metric g of any one of the three types and V a vector field (local or global) of M such that

$$\pounds_V g = 2\sigma g \tag{1.1}$$

where $\pounds_V$ is the Lie-derivative operator and σ is a function on M. Above equation is known as conformal Killing equation and the symmetry vector V is called a conformal Killing vector, briefly denoted by CKV. If σ is non-constant, then, V is called a proper CKV. In particular, V is homothetic or Killing according as σ is a no-zero constant or zero. The set of all proper CKV fields and all Killing vector fields on M form a finite dimensional Lie algebra.

The purpose of this article is to present a survey of research done on the geometry and physics of Riemannian, semi-Riemannian, in particular, Lorentzian and lightlike manifolds (M, g) having a metric symmetry defined by the equation (1.1). We collect the results of the two main symmetries, namely, Killing and conformal Killing and their two closely related sub-symmetries, called Affine Killing and Affine conformal Killing symmetries. This approach will help the reader to better understand the differences, similarities and relations between these two symmetries, with respect to their use in geometry and physics. A sketch of the proof of the most important results is given along with references for their link with several other related results.

The subject matter of metric symmetries is very wide and can not be covered in one review paper. For this reason we have provided a large number of references for more related results.

2. Riemannian and semi-Riemannian metric symmetries

Given a smooth manifold M, the group of all smooth transformations of M is a very large group. This leads to the study of those transformations of M which leave a certain physical/geometric quantity invariant. Related to the focus of this paper we let (M, g) a real n-dimensional smooth Riemannian or semi-Riemannian manifold. A diffeomorphism $\phi : M \to M$ is called an isometry of M if it leaves invariant the metric tensor g. This means that

$$g(\phi_\star X, \phi_\star Y) = g(X, Y), \qquad \forall X, Y \in \mathcal{X}(M),$$

where $\phi_\star$ is the differential (tangent) map of ϕ and $\mathcal{X}(M)$ denotes the set of all tangent vector fields on M. Since each tangent mapping $(\phi_\star)_p$, at $p \in M$, is a linear isomorphism of $T_p(M)$ on $T_{\phi(p)}(M)$, it follows that ϕ is an isometry if and only if $(\phi_\star)_p$ is a linear isometry for any $p \in M$. The set of all isometries of M forms a group under composition of mappings. Myers and Steenrod [6] proved that the group of all isometries of a Riemannian manifold is a Lie group. For analogous results on semi-Riemannian manifolds see O'Neill [2, chapter 9]. The isometric symmetry is related to a local infinitesimal transformation group as follows:

Let V be a smooth vector field on M and $\mathcal{U}$ a neighborhood of each $p \in M$ with coordinate system (x^i). Let the integral curves of V, through any point q in $\mathcal{U}$, be defined on an open interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. For each $t \in (-\epsilon, \epsilon)$ define an isometric map ϕ_t on $\mathcal{U}$ such that for q in $\mathcal{U}$, $\phi_t(q)$ is on the integral curve of V through q. Then, V generates a local 1-parameter group of infinitesimal transformations $\phi_t(x^i) = x^i + tV^i$ and we have

$$\partial_k(x^i + tV^i)\, \partial_m(x^j + tV^j)\, g_{ij}(x + tV) = g_{km}$$

which, after expanding $g_{ij}\,(x+tV)$ up to first order in t, yields to

$V^i\,\partial_i\,g_{jk}+\partial_j\,(V^i)\,g_{ik}+\partial_k(V^i)g_{ji}=0$

Using the Lie derivative operator $\pounds_V$, the above equation can be rewritten as

$\pounds_V g_{ij}=\nabla_i V_j+\nabla_j V_i=0$

where $V_i=g_{ij}V^j$ is the associated 1-form of V and ∇ is the Levi-Civita connection on M. Above Killing equations were named after a German mathematician Wilhelm Karl Joseph Killing [7] who made important contributions to the theories of Lie algebras, Lie groups, and non-Euclidean geometry. A simple example is a vector field on a circle that points clockwise and has the same length at each point is a Killing vector field since moving each point on the circle along this vector field just rotates the circle. For an n-dimensional Euclidean space, there exist $\frac{n(n+1)}{2}$ independent Killing vector fields. In general, any Riemannian or semi-Riemannian manifold which admits maximum Killing vector fields is called a manifold of constant curvature. This section contains important results on compact Riemannian, Kählerian, contact and semi-Riemannian manifolds.

2.1. Riemannian manifolds

The following divergence theorem is used in proofs of some results on the existence or non-existence of Killing and affine Killing vector fields.

Theorem 1 *Let (M,g) be a compact orientable Riemannian manifold with boundary ∂M. For a smooth vector field V on M, we have*

$$\int_M div\,V\,d\,v=\int_{\partial M} g(N,V)\,d\,S,$$

where N and $d\,S$ are the unit normal to ∂M and its surface element and $d\,v$ is the volume element of M.

Consider an n-dimensional Riemannian manifold (M,g) without boundary, that is, $\int_M divV=0$ holds for a smooth vector field V on M. Let V be a Killing vector field of (M,g), i.e.,

$$\pounds_V g=0 \quad \text{or} \quad \pounds_V g_{ij}=\nabla_j V_i+\nabla_i V_j=0, \quad (1\le i,j\ge n) \tag{2.1}$$

where ∇ denotes a symmetric affine connection on M. We start with the following fundamental theorem on the existence of a Killing vector field:

Theorem 2 Bochner [8]. *If Ricci tensor of a compact orientable Riemannian manifold (M,g), without boundary, is negative semi-definite, then a Killing vector field V on M is covariant constant. On the other hand, if the Ricci tensor on M is negative definite, then a Killing vector field other than zero does not exist on M.*

Bochner proved this theorem by assuming that V is a gradient of a function and a result of Watanabe [9] which states "$\int_M[Ric(V,V)-|\nabla V|^2]=0$ *if V is Killing*". Several other results on the geometry of compact Riemannian manifolds, without boundary, presented in Yano [10,11] are consequences of above result of Bochner.

Remark 3 Recall from M. Berger [12] that all known examples of compact Riemannian manifolds, with positive sectional curvature carry a positively curved metric with a continuous Lie group as its group of isometries. Thus, they carry a non-trivial Killing vector field. Moreover, such a Killing vector is singular at least at one point if the manifold is even dimensional. There are examples of odd dimensional closed positively curved Riemannian manifolds carrying non-singular Killing vector fields. A simple case is the 3-sphere S^3 which admits 3 pointwise linearly independent Killing vector fields while no two of them commute.

Killing symmetry has another closely associated symmetry, with respect to a symmetric affine connection ∇ on a non-flat Riemannian or semi-Riemannian manifold (M, g), defined as follows:

A vector field V on (M, g) is called affine Killing if $\pounds_V \nabla = 0$. To interpret this relation with respect to the metric g we split the tensor $\nabla_i V_j$ (see Killing equation (2.1)) into its symmetric and anti-symmetric parts as follows:

$$\nabla_i V_j = K_{ij} + F_{ij}, \quad (K_{ij} = K_{ji},\ F_{ij} = -F_{ji}). \tag{2.2}$$

Then, it follows that K_{ij} is covariant constant, i.,e,

$$\nabla_k K_{ij} = 0. \tag{2.3}$$

From equations (2.2) and (2.3) we deduce that V is affine Killing if and only if

$$\pounds_V g_{ij} = 2K_{ij}. \tag{2.4}$$

K_{ij} is called a proper tensor if it is different than the metric tensor g_{ij} of M and then V is called a proper affine Killing vector field. In general, for an n-dimensional manifold (M, g), the existence of a proper K_{ij} has its roots back in 1923, when Eisenhart [13] proved that a Riemannian M admits a proper K_{ij} if and only if M is reducible. This means that M is locally a product manifold of the form $(M = M_1 \times M_2\,,\ g = g_1 \oplus g_2\ )$ and there exists a local coordinate system in terms of which the distance element of g is given by

$$ds^2 = g_{ab}(x^c)\, dx^a\, dx^b + g_{AB}(x^C)\, dx^A\, dx^B,$$

where $a, b, c = 1, \ldots, r\,, A, B, C = r+1, \ldots, n$ and $1 \le r \le n$. Thus, an irreducible Riemannian manifold admits no proper affine Killing vector field.

Observe that the Killing equation (2.1) implies that the condition $\pounds_V \nabla = 0$ holds if V is Killing. However, not every affine Killing vector field is Killing. For example, it was shown in [14] that a non-Einstein conformally flat Riemannian manifold can admit an affine vector field for which K_{ij} is a linear combination of the metric tensor and the Ricci tensor. This result also holds for any non-recurrent, non-conformally flat and non-Einstein manifold which is conformally recurrent with a locally gradient recurrent vector [14]. Thus, affine vector fields in such spaces are proper since they are neither Killing nor homothetic. Also, see Subsections 2.3 and 3.1 for some examples of a proper affine Killing vector.

To find a class of Riemannian manifolds for which an affine Killing symmetry is Killing, Yano proved the following result:

Yano [11]. An affine Killing vector field on a compact orientable Riemannian manifold, without boundary, is Killing.

The proof is easy since V affine Killing implies $divV$ is constant on M and, in particular, $divV = 0$ if M is without boundary which implies V is Killing.

2.2. Kähler manifolds

A C^∞ real Riemannian manifold (M^{2n}, g) is called a Hermitian manifold if

$$J^2 = -I, \quad g(J\,X,\, J\,Y) = g(X,\, Y), \qquad \forall X, Y \in \mathcal{X}(M),$$

where J is a tensor field of type (1, 1) of the tangent space $T_p(M)$, at each point p of M, I is the identity morphism of $T(M)$ and $\mathcal{X}(M)$ denotes the set of all tangent vectors fields on M. The fundamental 2-form Ω of M is defined by

$$\Omega(X,\, Y) = g(X,\, JY), \qquad \forall X, Y \in \mathcal{X}(M).$$

(M, g, J) is called a Kähler manifold if Ω is closed. A vector field V on a Kähler manifold (M, g, J) is analytic if $\pounds_V J = 0$. It is easy to show that if V is analytic (also called holomorphic) on a Kähler manifold , then so is JV. Using this, one can easily show that if V is a Killing vector field on a compact Kähler manifold, then, JV is an analytic gradient vector.

Yano [15]. (a) In a compact Kähler manifold an analytic divergence free vector field is Killing.
(b) A Killing vector field on a compact Kähler manifold is analytic
(a) follows from $divV = 0$ and an integral formula (1.14) in [11, p. 41]. Then, (b) follows easily.

Sharma [16]. An affine Killing vector field V in a non-flat complex space form $M(c)$ is Killing and analytic.

Sharma first proved that the only symmetric (anti-symmetric) second order parallel tensor in a non-flat space form $M(c)$ is the Kählerian metric up to a constant multiple. Then, the proof follows from Yano [15].

Remark 4 Comparing Sharma's result with the part (b) of Yano's [15] result, observe that Sharma assumed V affine Killing and proved it to be Killing and analytic in $M(c)$ (not necessarily compact), whereas Yano required $M(c)$ to be compact (not necessarily of constant holomorphic sectional curvature).

In [17, pages 176 – 178] the reader can find other types of metric and curvature symmetries of Kähler manifolds, as a consequence of above two results.

2.3. Contact manifolds

A $(2n+1)$-dimensional differentiable manifold M is called a contact manifold if it has a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a given contact form η, there exists a unique global vector field ξ, called the characteristic vector field, satisfying

$$\eta(\xi) = 1, \qquad (d\,\eta)(\xi,\, X) = 0, \qquad \forall\, X \in \mathcal{X}(M).$$

A Riemannian metric g of M is called an associated metric of the contact structure if there exists a tensor field ϕ, of type (1, 1) such that

$$\begin{aligned} d\,\eta(X,\, Y) &= g(X,\, \phi Y), \qquad g(X,\, \xi) = \eta(X), \\ \phi^2\,(X) &= -\,X + \eta\,(X)\,\xi, \qquad \forall\, X,\, Y \in \mathcal{X}(M). \end{aligned}$$

These metrics can be constructed by the polarization of $d\,\eta$ evaluated on a local orthonormal basis of the tangent space with respect to an arbitrary metric, on the $2n$- dimensional contact sub bundle D of M. The structure $(\phi,\, \eta,\, \xi,\, g)$ on M is called a contact metric structure and its associated manifold is called a contact metric manifold which is orientable and odd dimensional. The contact metric structure is called a K-contact structure if its global characteristic vector field ξ is Killing. M has a normal contact structure if

$$N_\phi + 2\,d\,\eta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor field of ϕ. A normal contact metric manifold is called a Sasakian manifold which is also K-contact but the converse holds only if $dim(M) = 3$. The global characteristic Killing vector field ξ of a K-contact manifold has played a key role in the contact geometry. For details, see a complete set of Sasaki's works cited in Blair [18]

On the existence of a proper affine Killing vector field in contact geometry, we have the following non-trivial example:

Example 5 Let (M^{2n+1}, g) be a contact metric manifold such that

$R(X,Y)\xi = 0, \quad \forall\, X,\, Y \in \mathcal{X}(M^{2n+1})$.

Blair [19] proved that (M^{2n+1}, g) is locally the product of a Euclidean manifold E^{n+1} and S^n. Using this result, in 1985, Blair-Patnaik [20] used a tensor field $K = h - \eta \oplus \xi$ on a contact metric structure (ϕ, ξ, η, g) of M, where $h = \frac{1}{2}\pounds_\xi \phi$ is the self-adjoint trace-free operator and proved that

$$R(X,Y)\xi = 0 \quad \text{is equivalent to} \quad \nabla K(X,Y) = 0. \tag{2.5}$$

They also proved that if $R(\xi, X)\xi = 0$, there exists an affine connection annihilating K such that the curvature property is preserved. Thus, there exists a proper affine Killing vector field V of above described locally product contact metric manifold (M^{2n+1}, g), defined by $\pounds_V g_{ij} = 2K_{ij}$.

Other than above isolated example, the present author is not aware of any more case of proper affine Killing vector field in contact geometry. On the other hand, in [21] Sharma has proved the following two results

• *On a K-contact manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

• *An affine Killing vector field on a compact K-contact manifold without boundary is Killing.*

Then, in another paper [22], Sharma generalized above first result as follows:

• *Let M be a contact metric manifold whose ξ-sectional curvature $K(\xi, X)$ is nowhere vanishing and is independent of the choice of X. Then a second order parallel tensor on M is a constant multiple of the associated metric tensor.*

Remark 6 In [17, pages $182-185$] and [21,22] the reader can find several results on other types of metric and curvature symmetries of contact manifolds.

Now we quote the following two results involving manifolds with boundary.

Theorem 7 Yano-Ako [23] *A vector field V on a compact orientable manifold (M, g), with compact orientable boundary $\partial\, M$, is Killing if and only if*

(1) $\triangle\, V + Q\, V = 0 \quad , \quad div\, V = 0$ *on* M *and*

(2) $(\pounds_V g)(V, N) = 0$ *on* $\partial\, M$,

where Q is the $(1,1)$ tensor associated to the Ricci tensor of M.

For proof of above result and some side results on M with boundary, see Yano [11, pp. 118-120].

*Ü*nal [24] has proved a similar result for semi-Riemannian manifolds with boundary and subject to the following geometric condition:

For a semi-Riemannian M, the validity of divergence theorem is not obvious due to the possible existence of degenerate metric coefficient $g_{ii} = 0$ for some index i. Thus the boundary ∂M may become degenerate at some of its points or it may be a lightlike hypersurface of M. In both these cases, there is no well defined outward normal. *Ü*nal [24] studied this problem as follows:

Let M be a semi-Riemannian manifold with boundary ∂M (possibly $\partial M = \phi$). Its induced tensor $g_{\partial M}$ on ∂M is also symmetric but not necessary a metric tensor as it may be degenerate at some or all points of ∂M. Let ∂M_+ , ∂M_- and ∂M_0 be the subsets of points where the non-zero vectors orthogonal to ∂M are spacelike, timelike and lightlike respectively. Thus,

$$\partial M = \partial M_+ \cup \partial M_- \cup \partial M_0$$

where the three subsets are pairwise disjoint. Now we quote the following result:

Theorem 8 Ünal [24] *Let (M, g) be a compact orientable semi-Riemannian manifold with boundary ∂M such that its lightlike part ∂M_0 has measure zero in ∂M. Then, a vector field V on M is Killing if and only if*

(1) $\triangle V + QV = 0 \quad , \quad div\, V = 0$ *on M and*

(2) $(\pounds_V g)(V, N) = 0$ *on* $\partial M' = \partial M_+ \cup \partial M_-$,

where Q is the $(1, 1)$ tensor associated to the Ricci tensor of M, N is the unit normal vector field to ∂M induced on $\partial M'$ and all eigenvalues of $\pounds_V g$ are real.

Since the measure of lightlike ∂M_0 vanishes in ∂M, the proof of above result is exactly as in the case of Theorem 7 of Yano-Ako [23] and the use of following Gauss theorem which is also valid for any semi-Riemannian manifold.

Theorem 9 (Gauss) *Let M be a compact orientable semi-Riemannian manifold with boundary ∂M. For a smooth vector field V on M, we have*

$$\int_M (div\, V)\epsilon = \int_{\partial M} i_V \epsilon,$$

where $\epsilon = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n$ is the volume element on M and $g = det(g_{ij})$ with respect to a suitable local coordinate system $(x^1, \ldots, x^n)$. Here i denotes the operator of inner product.

2.1. Conformal Killing and affine conformal symmetries

Recall from the equation (1.1) that a vector field V of a Riemannian or semi-Riemannian manifold (M^n, g) is a conformal Killing vector field if $\pounds_V g = 2\rho g$ for some function ρ of M. To the best of our recollection, this conformal Killing equation appeared in a 1903 paper of Fubini [25] who studied the properties of infinitesimal conformal transformations of a metric space. Since then, the subject matter on conformal Killing vector (CKV) fields is indeed very wide both in geometry and physics. Here we present main results on the existence or non-existence of CKV fields and one of its closely related symmetry.

We first link Bochner's Theorem 2 for Killing vector field with the following general existence theorem for a conformal Killing vector field:

Theorem 10 Yano [10] *If the Ricci tensor of a compact orientable Riemannian manifold (M, g), without boundary, is non-positive, then, a CKV field V has a vanishing covariant derivative (hence Killing). If the Ricci tensor is negative-definite, then, there does not exist any CKV field on M.*

Yano proved above theorem by assuming that V is a gradient of a function and used an integral formula [11, page 46] which states

$$\int_M [Ric(V, V) - |\nabla V|^2 - \frac{n-2}{n}(\delta V)^2] dV = 0$$

if V is a CKV, where $\rho = \frac{1}{n}\delta V$. See in [11] for several other results coming from Yano's above theorem, involving conditions on the curvature.

In 1971, Obata proved the following result on "*Conformal transformations*":

Theorem 11 Obata [26] *If the group of conformal transformations of a compact Riemannian manifold is noncompact, then this manifold is conformally diffeomorphic to the standard sphere.*

This theorem was extended to the noncompact case by J. Ferrand [27] in 1994. Following results are direct consequences of the above theorem of Obata:

Yano-Nagano [28]. A complete connected Einstein manifold M (dimension $n \geq 2$), admitting a proper CKV field, is isometric to a sphere in an $(n+1)$-dimensional Euclidean space.

Yano [29]. In order that a compact Riemannian manifold M (dimension $n > 2$), with constant scalar curvature $r =$ constant and admitting proper CKV field (with conformal function σ) to be isometric to a sphere, it is necessary and sufficient that

$$\int_M (Ric - \frac{r}{n}\, g\,)(grad\, \sigma\,,\, grad\, \sigma)\, dv = 0.$$

Lichnerowicz [30]. Let a compact Riemannian manifold (M, g) admit a proper CKV field, with conformal function σ, such that one of the following holds:

(1) The 1-form associated with V is exact,

(2) $grad\, \sigma$ is an eigenvalue of the Ricci tensor with constant eigenvalues,

(3) $L_V\, Ric = f\, g$, for some smooth function f.

Then M is isometric to a sphere.

Yano [11]. (1) If (M, g) is complete, of dimension $n > 2$, with $r =$constant > 0, and if it admits a proper CKV field, with conformal function σ, then

$$\sigma^2\, r^2 \leq n\,(n-1)^2\, |\nabla\, \nabla\, \sigma|^2,$$

and equality holds if and only if M is isometric to a sphere.

(2) If a complete Riemannian manifold (M, g), of dimension $n > 2$, with scalar curvature r admits a proper CKV field V that leaves the length of the Ricci tensor Ric invariant, i.e., $V(|Ric|) = 0$, then, M is isometric to a sphere.

We refer Yano [11, pp 120-124] for results on CKV fields in M with boundary.

Critical Remark. In the world of Mathematical Science and Engineering, the Stokes and divergence theorems are like founding pillars for a large variety of practical (small and or big) problems. I believe this was the main motivation that *Ü*nal [24]'s Theorem 8 appeared in 1995 to use those founding theorems in semi-Riemannian geometry. However, unfortunately, the idea of this reference has not yet been picked by the research community to show a similar use of Stokes and divergence theorems (even with essential restrictions) for semi-Riemannian manifolds. There is a need to take a step in this direction.

Since there is no generalization to the Hopf-Rinow theorem for the semi-Riemannian case, related to problems with metric symmetry it remains an open question to verify the above quoted results when the Riemannian metric is replaced by a metric of arbitrary signature.

On the other hand, in recent years a systematic study of timelike Killing and conformal Killing vector fields on Lorentzian manifolds has been developed by using Bochner's technique for which we refer the works of Romero-S*á*nchez [32] and Romero [31]. In case of conformal Killing vector fields in general semi-Riemannian manifolds, we refer two papers of K*ü*hnel-Rademacher [33,34].

3. Metric symmetries in spacetimes

Let (M, g) be an n-dimensional time-orientable Lorentzian manifold, called a spacetime manifold. This means that M is a smooth connected Hausdorff manifold and g is a time orientable Lorentz metric of normal hyperbolic signature $(- + \ldots +)$. For physical reason, we collect main results on Killing symmetry used in a 4-dimensional spacetime of general relativity. Later on we present some general results for n-dimensional ($n \geq 3$) compact time orientable Lorentzian manifolds.

Consider the following form of Einstein field equations:

$$R_{ij} - \frac{1}{2} r g_{ij} = T_{ij}, \quad (i, j = 1, \dots 4),$$

where T_{ij}, R_{ij} and r are the stress-energy tensor, the Ricci tensor and the scalar curvature respectively. T_{ij} is said to obey the mixed energy condition if at any point x on any hypersurface, (i) the strong energy condition holds, i.e., $T_{11} + T^i_i|_x \geq 0$ and (ii) equality in (i) implies that all components of T are zero. T is said to obey the dominant energy condition if in any orthonormal basis the energy dominates the other components of T_{ij}, i.e., $T_{11} \geq |T_{ij}|$ for each i, j. Since the Einstein field equations are a complicated set of non-linear differential equations, most explicit solutions (see Kramer et al. [35]) have been found by using Killing or homothetic symmetries. This is due to the fact that these symmetries leave the Levi-Civita connection, all the curvature quantities and the field equations invariant.

Considerable work is available to show that not any arbitrary time-orientable Lorentzian manifold may be physically important as compared to the choice of a prescribed model of spacetimes. Related to the metric symmetries, following is a widely used model of spacetimes:

A spacetime (M, g) is called globally hyperbolic [3] if there exists an embedded spacelike 3-manifold Σ such that every endless causal curve intersects Σ once and only once. Such a hypersurface Σ, if it exists, is called a Cauchy surface. If M is globally hyperbolic, then (a) M is homeomorphic to $R \times S$, where S is a hypersurface of M, and for each t, $\{t\} \times S$ is a Cauchy surface, (b) if S' is any compact hypersurface without boundary, of M, then S' must be a Cauchy surface. It is obvious from above that Minkowski spacetime is globally hyperbolic. In the following we present a characterization result of Eardley-Isenberg-Marsden-Moncrief [36] on the existence of Killing or homothetic vector field in globally hyperbolic spacetimes.

Theorem 12 [36] *Let (M, g) be a globally hyperbolic space-time which*

(1) satisfies the Einstein equations for a stress energy tensor T obeying the mixed energy and the dominant energy conditions.

(2) Admits a homothetic vector field V of g.

(3) Admits a compact hypersurface Σ of constant mean curvature.

Then, either (M, g) is an expanding hyperbolic model with metric

$$ds^2 = e^{\lambda t}(-dt^2 + h_{ab}\, dx^a\, dx^b), \tag{3.1}$$

with $h_{ab}\, dx^a\, dx^b$ a 3-dimensional Riemannian metric of constant negative curvature on a compact manifold and T vanishing, or V is Killing.

Sketch of proof. According to a result by Geroch [37] we know that if a globally hyperbolic spacetime (M, g) satisfies the vacuum Einstein equations, i.e., T vanishes, then g may be completely determined from a set of Cauchy data specified on (Σ, γ) or if M satisfies the Einstein equations coupled to a well-posed hyperbolic systems of matter equations, then the coupled system has the same property, where γ is the induced 3-metric of Σ. Using this property, above theorem was proved within the environment of 3-dimensional compact spacelike hypersurface Σ of (M, g). By hypothesis, if the mean curvature c of Σ is zero, then, V is Killing and so the theorem is obvious. If $c \neq 0$, then, it can be proved that Σ is totally umbilical in M and is of negative constant curvature. Then, it follows from a theorem of Bochner [8] that the standard hyperbolic metric admits no non-zero global Killing vector field. Finally, it is easy to show that (for $c \neq 0$) the vacuum spacetime M is an expanding hyperbolic model as presented in the form (3.1), which completes the proof.

As an application of above theorem, consider the Einstein-Yang-Mills equations [38], with the gauge group chosen to be a compact Lie group. The Lie-algebra-valued Yang-Mills field F has the components

$$F_{ij} = D_i\, A_j - D_j\, A_i + [A_i, A_j],$$

where A_i and D_i are the gauge potential and the spacetime covariant derivative operator with respect to g, respectively. The Einstein-Yang-Mills equations are

$$\begin{aligned} R_{ij} - \frac{1}{2}\, r\, g_{ij} &= 8\, \pi\, T_{ij}, \\ T_{ij} &= \frac{1}{4} F_{km} F^{km} g_{ij} - F_{ik} F^k_j, \\ D^i\, F_{ij} &+ [A^i, F_{ij}] = 0. \end{aligned}$$

Above equations satisfy mixed and dominant energy conditions. It is easy to show that if the condition (1) of Theorem 12 is replaced by [(1) satisfies the Einstein-Yang-Mills equations], then one can show that either M is expanding hyperbolic model with metric (3.1) and field $F \equiv 0$ everywhere or V is Killing.

Another application is of a massless scalar field ψ coupled to gravity for which the Einstein-Klein-Gordon equations[39] are Einstein equations with

$$T_{ij} = (D_i\, \psi)(D_j\, \psi) - \frac{1}{2}\, g_{ij} (D^k\, \psi)(D_k\, \psi), \quad D^i\, D_i\, \psi = 0.$$

In this case, since T_{ij} does not satisfy the mixed energy condition, we quote the following theorem (proof is common with the proof of above theorem).

Theorem 13 [36] *Let (M, g) be a globally hyperbolic spacetime which*

(1) satisfies the Einstein-Klein-Gordon equations,

(2) admits a homothetic vector field V of g and

(3) admits a compact hypersurface of constant mean curvature.

Then, either M is an expanding hyperbolic model with metric (3.1) and ψ is constant everywhere, or V is Killing.

3.1. Affine Killing vector fields in spacetimes

We know from Subsection 2.1 that a vector field V of a semi-Riemannian manifold (M, g) is an affine Killing vector field if

$$\pounds_V\, g_{ij} = 2\, K_{ij}\,, \quad K_{ij;k} = 0,$$

where K_{ij} is a covariant constant second order symmetric tensor. V is proper affine if K_{ij} is other than g_{ij}. Eisenhart's [13] Riemannian result (see Subsection 2.1) was generalized by Patterson [40], in 1951, showing that a semi-Riemannian (M, g) admitting a proper K_{ij} is reducible if the matrix of K_{ij} has at least two distinct characteristic roots at any point of M. Since then, a general characterization of affine Killing symmetry (known as affine collineation symmetry) remains open. However, for a 4-dimensional spacetime M, this problem has been completely resolved (see Hall and da Costa [41]). Global study requires the spacetime to be simply connected (which means that any closed loop through any point can be shrunk continuously to that point), and for local considerations one may restrict to a simply connected region. We now know from [41] that if a simply connected spacetime (M, g) admits a global, nowhere zero, covariant constant proper K_{ij}, then one of the following three possibilities exist:

(a) There exists locally a timelike or spacelike, nowhere zero covariant constant vector field ξ such that $K_{ij} = \eta_i \, \eta_j, \quad \eta_i = g_{ij} \, \xi^j$ and M is locally decomposable into $(1+3)$ spacetime.

(b) There exists locally a null, nowhere zero, covariant constant vector field ξ such that K_{ij} is as in (a) but (M, g), in general, is not reducible.

(c) M is locally reducible into a $(2+2)$ spacetime and no covariant constant vector exists unless it decomposes into $(1+1+2)$ spacetime (a special case of (b)). For the latter case, there exist two such proper covariant constant tensors of order 2.

In another paper, Hall et al [42] has proved that the existence of a proper affine Killing symmetry eliminates all vacuum spacetimes except the plane waves, all perfect fluids when the pressure $\neq$ density and all non-null Einstein Maxwell fields except the $(2+2)$ locally decomposable case. Hence, affine Killing symmetry has very limited use in finding exact solutions. We end this section with two examples of spacetimes admitting proper affine Killing vector fields.

Example 14 Consider the Robertson-Walker metric in spherical coordinates (t, r, θ, ϕ) with

$$ds^2 = dt^2 - S^2(t) \, ((1 - \mathcal{K} \, r^2)^{-1} \, dr^2 + r^2 \, d\theta^2 + r^2 \, sin^2\theta \, d\phi^2),$$

where $\mathcal{K} = 0, \pm 1$. Let $V^i = \lambda(t) \, \delta^i_t$ be a timelike vector parallel to the fluid flow vector $u^i = \delta^i_t$. using affine Killing equation $V_{i;j} + V_{j;i} = 2 \, K_{ij}$, we obtain

$$\begin{aligned} V_{i;j} &= K_{ij} = \delta^t_i \, \delta^t_j \, \dot{\lambda} \\ &- \lambda \, S \, \dot{S} [\delta^r_i \, \delta^r_j (1 - \mathcal{K} r^2)^{-1} + \delta^\theta_i \, \delta^\theta_j \, r^2 + \delta^\phi_i \, \delta^\phi_j \, r^2 \, \sin^2\theta]. \end{aligned} \tag{3.2}$$

Since K_{ij} is covariant constant, $V_{(i;j);k} = 0$. Calculating this later equation, we get $\lambda \, \dot{S} - S \, \dot{\lambda} = 0$ and $\ddot{\lambda} = 0$. Thus, we obtain

$$\lambda = a \, S(t) \qquad S = b \, t + c, \tag{3.3}$$

for some constants a, b and c. Thus, V is a timelike vector field parallel to u such that a proper K_{ij} is given by (3.2) and λ and S are related by (3.3).

Example 15 The Einstein static universe, which is simply connected and complete manifold $M = R^1 \times S^3$, with the metric

$$ds^2 = -dt^2 + dr^2 + \sin^2 r (d\theta^2 + \sin^2\theta \, d\phi^2)$$

admits [41] an 8-dimensional transitive Lie group of affine transformations generated by the global proper affine vector field $V = t\partial_t$.

3.2. Spacetimes with conformal Killing symmetry

Although the use of CKV is not desirable in finding exact solutions (as CKV's do not leave the Einstein tensor invariant), nevertheless, now we know quite a number of physically important results (including exact solutions) using conformal symmetry. To review the main latest results on conformal symmetry, we choose one of the widely used model of $(1+3)$-splitting (Arnowitt-Deser-Misner [43]) 4-dimensional spacetime (M, g). This assumes a thin sandwich of M evolved from a spacelike hypersurface Σ_t at a coordinate time t to another spacelike hypersurface Σ_{t+dt} at coordinate time $t+dt$ with metric g given by

$$g_{\alpha\beta}dx^\alpha dx^\beta = (-\lambda^2 + S^a S_a)dt^2 + 2\gamma_{ab}S^a dx^b dt + \gamma_{ab}dx^a dx^b$$

where λ is the lapse function, S is the shift vector, $x^0 = t, x^a (a = 1, 2, 3)$ are spatial coordinates and γ_{ab} is the 3-metric on spacelike slice Σ. This is known as ADM model which admits a CKV field. In 1986, Maartens-Maharaj [44] proved that Robertson-Walker spacetimes (which provide a satisfactory cosmological ADM model) admit a G_6 of Killing vectors and a G_9 of conformal Killing vector fields. By definition, a group G_r of isometric or conformal motions has r Killing or conformal Killing vectors as generators, respectively. We need the following constraint and conformal evolution equations for the ADM model:

Denote arbitrary vector fields of Σ by X, Y, Z, W, and the timelike unit vector field normal to Σ by N. Then the Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \bar{\nabla}_X N = A_N X$$

where A_N is the shape operator of Σ defined by $B(X, Y) =< A_N X, Y >$ ($<, >$ is the inner product with respect to the metric γ of Σ and the spacetime metric g), $\bar{\nabla}, \nabla$ the Levi-Civita connections of g, γ respectively and B is the second fundamental form. The Gauss and Codazzi equations are

$$\begin{aligned} < \bar{R}(X, Y)Z, W > &= < R(X, Y)Z, W > + B(Y, Z)B(X, W) \\ &= -B(X, Z)B(Y, W) \\ < \bar{R}(X, Y)N, Z > &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \end{aligned}$$

where $\bar{R}$ and R denote curvature tensors of g and γ respectively. It is straightforward to show that the following relation holds

$$\begin{aligned} \bar{Ric}(X, Y) + < \bar{R}(N, X)Y, N > &= Ric(X, Y) + \tau < A_N X, Y > \\ &= - < A_N X, A_N Y > \\ \bar{Ric}(X, N) &= (div A_N)X - X\tau \end{aligned}$$

where $\bar{Ric}$ and Ric are the Ricci tensors of g and γ respectively, and $\tau = Tr.A_N = 3$ times the mean curvature of Σ. Let the Einstein's field equations be of the form $\bar{Ric} - \frac{\bar{r}}{2}g = T$, where $\bar{r}$ and T are the scalar curvature and the energy-momentum tensor respectively. Following are the constraint equations

$$\begin{aligned} r + \frac{2\tau^2}{3} - |L|^2 &= 2T(N, N), \quad L = A_N - \frac{\tau}{3}I, \\ (divL)X - \frac{2}{3}X\tau &= T(X, N) \end{aligned}$$

where r is the scalar curvature of γ, $||$ the norm operator with respect to γ. Assume that (M, g) admits a CKV field V, i.e., $\pounds_V g = 2\sigma g$. Decompose V along Σ as $V = \xi + \rho N$, where ξ is the tangential component of V. A simple calculation using all the above equations provides the following evolution equation:

$$(\pounds_\xi \gamma)(X, Y) = 2\sigma\gamma(X, Y) - 2\rho < LX, Y > - \frac{2\rho\tau}{3}\gamma(X, Y).$$

$$
\begin{aligned}
(\pounds_\xi L)X &= -(\nabla_X D\rho - \frac{\nabla^2\rho}{3}X) - \rho(TX - \frac{T^i_i}{3}X) \\
&\quad + (\rho\tau - \frac{\sigma}{2})LX + \rho(QX - \frac{r}{3}X)
\end{aligned}
$$

$$
\pounds_\xi \tau = \sigma(3N - \tau) - \nabla^2\rho + \rho[\frac{\tau^2}{3} + |L|^2 + \frac{1}{2}(T^i_i + T(N,N))].
$$

Here Q is the Ricci operator of γ, and $\nabla^2 = \nabla^a\nabla_a$ (a summed over $1,2,3$). Above Evolution equations were first derived in [36] through a different approach using B. Berger's [45] condition that sets the evolution vector field equal to V.

Theorem 16 Sharma [46] *Let (M,g) be an ADM spacetime solution of Einstein's field equations admitting a CKV field V and be evolved by a complete spacelike hypersurface Σ such that (a) Σ is totally umbilical in M (b) the normal component ρ of V is non-constant on Σ, and (c) the normal sectional curvature of M is independent of the tangential direction at each point of Σ. Then Σ is conformally diffeomorphic to (i) a 3-sphere S^3, or (ii) Euclidean space E^3, or (iii) hyperbolic space H^3, or (iv) the product of a complete 2-dimensional manifold and an open real interval. If Σ is compact, then only (i) holds.*

Sharma's proof uses above constraint and evolutions equations with the condition that the normal sectional curvature $\bar{S}(N,X)$ of M at a point p with respect to a plane section spanned by a unit tangent vector X of Σ and the unit normal N, is independent of the choice of X. Note that the normal sectional curvature is defined as $< \bar{R}(N,X)N, X >$ (see [3, page 33]). This normal sectional curvature holds when M is Minkowski, de Sitter, anti-de Sitter, and Robertson-Walker spacetime.

Example 17 Consider the following generalized Robertson-Walker (GRW) spacetime as the warped product $(M = I \times_f \Sigma, g)$ defined by

$$
ds^2 = -dt^2 + (f(t))^2\gamma_{ab}dx^a dx^b,
$$

where I is the time line, (Σ,γ) is an arbitrary $3D$-Riemannian manifold and $f > 0$ is a warping function (see Alias-Romero-Sánchez [47]). They have shown that the normal curvature condition holds for this GRW-spacetime and each slice $t =$ constant is homothetic to the fiber Σ, and totally umbilical in (M,g).

As a consequence of the Theorem 16, following two results are easy to prove:

Duggal-Sharma [48]. (1) Let (M,g) be a ADM spacetime evolved out of a complete initial hypersurface Σ that is totally umbilical and has non-zero constant mean curvature. If (M,g) admits a closed CKV field V non-vanishing on Σ, then, either V is orthogonal to Σ and the lapse function is constant over Σ, or Σ is conformally diffeomorphic to E^3, or S^3, or H^3, or the product of an open interval and a 2-dimensional Riemannian manifold.

(2) Let a conformally flat perfect fluid solution (M,g) of the Einstein's equations be evolved out of an initial spacelike hypersurface Σ that is compact, has constant mean curvature, and is orthogonal to the 4-velocity. If (M,g) has a non-vanishing non-Killing CKV field V which is nowhere tangential to Σ, then, Σ is totally umbilical in M, and is of constant curvature. In the case when M is of constant negative curvature, V is orthogonal to Σ.

3.3. Spacetimes with affine conformal symmetry

An affine conformal symmetry is defined by a vector field V of (M, g) satisfying

$$\pounds_V g = \sigma g + K, \quad \nabla K = 0$$

where $K \neq g$ is a second order symmetric tensor and V is called an affine conformal Killing vector [49], denoted by ACV, which is CKV when K vanishes. If σ is constant, then, V is affine. Moreover, V is an ACV if and only if

$$\pounds_V \Gamma^k_{ij} = \delta^k_i \partial_j(\sigma) + \delta^k_j \partial_i(\sigma) - g_{ij}\sigma^k,$$

which is also known as "*conformal collineation symmetry*" generated by an ACV field V. Here Γ^k_{ij} are the Christoffel symbols. We state the main results on ACV (proved by Tashiro [50]) on the local reducibility of a Riemannian manifold (M, g). By local reducibility we mean that M is locally a product manifold.

(1) If M has constant scalar curvature and has a flat part, then an ACV on M is the sum of an affine and a CKV.

(2) If M has at least three parts and no part is locally flat, then an ACV on M is affine. If M is also complete, then the ACV is Killing.

(3) Let M has constant scalar curvature with no flat part. If M is irreducible or is the product of two irreducible parts whose scalar curvatures are signed opposite to each other, then, an ACV on M is a CKV. Otherwise, it is affine.

(4) A globally defined ACV on a Euclidean space is necessarily affine.

(5) A Riemannian manifold of constant curvature does not admit an ACV.

(6) An irreducible M admits no ACV which is not a CKV.

(7) If a locally reducible M has at least three parts, one of which is flat, then an ACV on M is sum of an affine vector and a CKV. If M is also complete, then the ACV is affine.

Remark 18 For a semi-Riemannian manifold, a general characterization of an ACV still remains open, although limited results are available in [49,51]. As an attempt to verify some or all results listed above, Mason and Maartens [51] constructed the following example which supports first part of the result (7).

Example 19 Let (M^4, g) be a Einstein static fluid spacetime with metric

$$ds^2 = -dt^2 + (1-r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

and the velocity vector $u^a = \delta^i_0$ ($i = 0, 1, 2, 3$). This spacetime admits a CKV

$$V^i_1 = (1-r^2)^{1/2}\{\cos t \, u^i - r \sin t \, \delta^i_1\}$$

and a proper affine vector $V^i_2 = t\, u^i$. Since the metric is reducible, it can be easily verified that a combination $V = V_1 + V_2$ is a proper ACV such that

$$\begin{aligned} V^i &= [t + (1-r^2)^{1/2} \, cos\, t]\, u^i - r(1-r^2)^{1/2} \, sin\, t\, \delta^i_1, \\ \sigma &= -(1-r^2)^{1/2} \, sin\, t, \qquad K_{ij} = -2\, t_{,i}\, t_{,b}. \end{aligned}$$

Now let (M^n, g) be a compact orientable semi-Riemannian manifold with boundary ∂M. The divergence theorem is not valid due to the possible degenerate part of ∂M. For this reason we call (M, g) a regular [49] semi-Riemannian manifold if we exclude the possible degenerate part in ∂M. Then, following is a characterization theorem for the existence of a proper ACV:

Theorem 20 Duggal [49] *A vector field V in a compact orientable regular semi-Riemannian manifold $(M,\, g)$, with boundary ∂M, is a proper ACV if and only if*

(a) $\int_{\partial M} (K - \frac{tr.H}{n}\, g)\,(V,\, N)\, ds \neq 0$

(b) $\mathcal{D}\, V = -\,(n-2)\, grad\, \sigma \in M, \quad \mathcal{D}\, V = Q\, V + \triangle\, V,$

where σ, K and H are the de-Rham Laplacian, affine conformal function, covariant constant tensor of type $(0, 2)$ and its associated $(1,\, 1)$ tensor respectively.

The reader will find several other side results in [17, Chapter 7] on the geometry and physics of affine conformal symmetry.

4. Compact time orientable Lorentzian manifolds

Recall that the famous Hopf-Rinow theorem maintains the equivalence of metric and geodesic completeness and, therefore, guarantees the completeness of all Riemannian metrics, for a compact smooth manifold, with the existence of minimal geodesics. Also, if this theorem holds, then, the Riemannian function is finite-valued and continuous. Unfortunately, for an indefinite metric, completeness is a more subtle notion than in the Riemannian case, since there is no satisfactory generalization to the Hopf-Rinow theorem for a semi-Riemannian manifold. There are some isolated cases satisfying metric and / geodesic completeness. For example, in 1973, Marsden [52] proved that "*every compact homogeneous semi-Riemannian manifold is geodesically complete*". For the case of Lorentzian manifolds, the singularity theorems (see Hawking-Ellis [39]) confirm that not all Lorentz manifolds are metric and / geodesic complete. Also, the Lorentz distance function fails to be finite and / or continuous for all arbitrary spacetimes [3]. It has been shown in Beem-Ehrlich's book [3] that the globally hyperbolic spacetimes turn out to be the most closely related physical spaces sharing some properties of Hopf-Rinow theorem. Now we know that timelike Cauchy completeness and finite compactness are equivalent and the Lorentz distance function is finite and continuous for this class of spacetimes.

We have seen in previous sections that metric symmetries have a key role in 4-dimensional paracompact globally hyperbolic spacetimes. In this section we let (M, g) be an n-dimensional $(n \geq 3)$ compact time orientable Lorentzian manifold. Recall that a compact manifold M admits a Lorentzian metric if and only if the Euler number of M vanishes. Considerable work has been done on the applications of null geodesics of compact (M, g) using a conformal Killing symmetry. Since, for Lorentzian metrics the compactness does not imply geodesic completeness, Romero-Sánchez [32] have proved that a compact Lorentzian manifold which admits a timelike CKV field yields to its geodesic completeness.

Let $C(s)$ be a curve in a Lorentzian manifold $(M,\, g)$, where s is a suitable parameter. A vector field V on C is called a Jacobi vector field if it satisfies the following Jacobi differential equation:

$$\nabla_{C'}\, \nabla_{C'}\, V = R(C',\, V)C',$$

where ∇ is a metric connection on M.

Definition 21 *We say that a point p on a geodesic $C(s)$ of M is conjugate to a point q along $C(s)$ if there is a Jacobi field along $C(s)$, not identically zero, which vanishes at q and p.*

From a geometric point of view, a conjugate point $C(a)$ of $p = C(0)$ along a geodesic C can be interpreted as an "almost-meeting point" of a geodesic starting from p with initial velocity $C'(0)$. In general relativity, since the relative position of neighboring events of a free falling particle C

is given by the Jacobi field of C, the attraction of gravity causes conjugate points, while the non attraction of gravity will prevent them. Although a physical spacetime is generally assumed to be causal (free of closed causal curves), all compact Lorentzian manifolds are acausal, i.e., they admit closed timelike curves. See [3, chapters 10 and 11, Second Edition] in which they have done extensive work on conjugate points along null geodesics of a general Lorentzian manifold which may be causal or acausal. We need the following notion of null sectional curvature [3].

Let $x \in (M,\, g)$ and ξ be a null vector of T_xM. A plane H of T_xM is called a null plane directed by ξ if it contains ξ, $g_x(\xi,\, W) = 0$ for any $W \in H$ and there exists $W_o \in H$ such that $g_x(W_o,\, W_o) \neq 0$. Then, the null sectional curvature of H, with respect to ξ and ∇, is defined as a real number

$$K_\xi\,(H) = \frac{g_x(R(W,\, \xi)\xi,\, W)}{g_x(W,\, W)},$$

where $W \neq 0$ is any vector in H independent with ξ (and therfore spacelike). It is easy to see that $K_\xi\,(H)$ is independent of W but depends in a quadratic fashion on ξ. The null congruence associated with a vector field V is defined by

$$C_VM = \{\xi \in TM : g(\xi,\, \xi) = 0,\ g(\xi,\, V_{\pi(\xi)}) = 1\},$$

where $\pi : TM \to M$ is the natural projection. C_KM is an oriented embedded submanifold of TM with dimension $2(n-1)$ and $(C_VM,\, \pi,\, M)$ is a fiber bundle with fiber type S^{n-2}. Therefore, for a compact M, C_VM will be compact. If a null congruence C_VM is fixed with respect a timelike vector field V, then one can choose, for every null plane H, the unique null vector $\xi \in C_VM \cap H$, so that the null sectional curvature can be thought as a function on null planes. This function is called the V-normalized null sectional curvature.

Gutiérrez-Palomo-Romero [53–55] have done following work on conjugate points along null geodesics of compact Lorentzian manifolds:

[53] Let $(M,\, g)$ be an n-dimensional $(n \geq 3)$ compact Lorentzian manifold that admits a timelike CKV field V. If there exists a real number $a \in (0,\, +\infty)$ such that every null geodesic $C_\xi : [0,\, a] \to M$, with $\xi \in C_VM$, has no conjugate points of $C_\xi(0)$ in $[0,\, a)$, then

$$Vol(C_VM,\, \hat{g}) \geq \frac{a^2}{\pi^2 n(n-1)} \int_{C_VM} \bar{Ric}\, d\mu_{\hat{g}}.$$

Equality holds if and only if M has V-normalized null sectional curvature $\frac{\pi^2}{a^2}$. Here $\hat{g}$ is the restriction to C_VM of the metric on the TM. $\bar{Ric}$ denotes the quadratic form associated with the Ricci tensor of M and $d\mu_{\hat{g}}$ is the canonical measure associated with $\hat{g}$.

[54] The authors used above result in proving several inequalities relating conjugate points along geodesics to global geometric properties. Also, they have shown some classification results on certain compact Lorentzian manifolds without conjugate points along its null geodesics.

[55] Let $(M^n,\, g)$ be a compact Lorentzian manifold admitting a timelike CKV field V. If $(M^n,\, g)$ has no conjugate points along its null geodesic, then

$$\int_M [\bar{Ric}(U) + S]\, h^n\ d\mu_g \leq 0,$$

where $h = [-g(V,\, V)]^{-1/2}$ so that $g(U,\, U) = -1$ with $U = hV$. Moreover, equality holds if and only if $(M,\, g)$ has constant sectional curvature $k \leq 0$. If V is a timelike Killing vector field, then

$$\int_M S\, h^n\ d\mu_g$$

and equality holds if and only if M is isomorphic to a flat Lorentzian n-torus up to a (finite) covering. In particular, U is parallel, the first Betti number of M is non-zero and the Levi-Civita connection of g is Riemannian.

Remark 22 Recall the following classical Hopf theorem [56] :

"*A Riemannian torus with no conjugate points must be flat.*"

As a Lorentzian analogue to Hopf theorem, Palomo and Romero [57] have recently proved the following result:

"*A conformally stationary Lorentzian tori with conjugate points must be flat.*"

On the other hand, in another paper Palomo and Romero [58] have obtained a sequence of integral inequalities for any ($n \geq 3$)-dimensional compact conformally stationary Lorentzian manifold with no conjugate points along its causal geodesics. The equality for some of them implies that the Lorentzian manifold must be flat.

5. Metric symmetries in lightlike geometry

Let (M, g) be an n-dimensional smooth manifold with a symmetric (0, 2) tensor field g. Assume that g is degenerate on TM, that is, there exists a vector field $\xi \neq 0$, of $\Gamma(TM)$, such that $g(\xi, v) = 0$, $\forall\, v \in \mathcal{X}(TM)$. The radical distribution of TM, with respect to g, is defined by

$$Rad\,TM = \{\xi \in \Gamma(TM)\,;\, g(\xi, v) = 0\,, \forall v \in \mathcal{X}(TM)\}\,.$$

such that $TM = Rad(TM) \oplus_{orth} S(TM)$, where $S(TM)$ is a non-degenerate complementary screen distribution of $RadTM$ in TM. Suppose $dim(Rad(TM)) = r \geq 1$. Then, $dim(S(TM)) = n - r$. As in case of semi-Riemannian manifolds, a vector field V on a lightlike manifold (M, g) is said to be a Killing vector field if $\pounds_V g = 0$. A distribution D on M is called a Killing distribution if each vector field belonging to D is a Killing vector field. Due to degenerate g on M, in general, there does not exist a unique metric (Levi-Civita) connection for M which is undesirable. Killing symmetry has the following important role in removing this anomaly:

Theorem 23 [59, page 49] *There exists a unique Levi-Civita connection on a lightlike manifold (M, g) with respect to g if and only if $Rad(TM)$ is Killing.*

Above result also holds if (M, g) is a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ for which $Rad(TM) = TM \cap TM^{\perp}$ (see [59, page 169].

We refer following two books [60,61] which include up-to-date information on extrinsic geometry of lightlike subspaces, in particular reference to a key role of Killing symmetry.

Physical Interpretation. Physically useful are the lightlike hypersurfaces of spacetime manifolds which (under some conditions) are models as black hole horizons (see Carter [62], Galloway [64] and other cited therein). To illustrate this use, let (M, g) be a lightlike hypersurface of a spacetime manifold $(\bar{M}, \bar{g})$. We adopt following features of the intrinsic geometry of lightlike hypersurfaces: Assume that the null normal ξ is not entirely in M, but, is defined in some open subset of $\bar{M}$ around M. This well-defines the spacetime covariant derivative $\bar{\nabla}\xi$, which, in general, is not possible if ξ is restricted to M as is the case of extrinsic geometry, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. Following Carter [63], a simple way is to consider a foliation of $\bar{M}$ (in the vicinity of M) by a family (M_u) so that ξ is in the part of $\bar{M}$ foliated by this family such that at each point in this region, ξ is a null normal to M_u for some value of u. Although the family (M_u) is not unique, for our purpose we can manage (with some reasonable condition(s)) to involve only those quantities which are independent of the choice of the foliation (M_u) once evaluated at, say, $M_{u=}$ constant. For simplicity, we denote by $M = M_u =$ constant. Then the metric g is simply the pull-back of the metric $\bar{g}$ of $\bar{M}$ to M, $g_{ij} = \underleftarrow{\bar{g}_{ij}}$, where an under arrow denotes the pullback to M. The "bending" of M in M is described by the *Weingarten map*:

$$\begin{aligned} \mathcal{W}_\xi : T_pM &\rightarrow T_pM \\ X &\rightarrow \bar{\nabla}_X \xi, \end{aligned} \tag{5.1}$$

that is, $\mathcal{W}_\xi$ associates each X of M the variation of ξ along X, with respect to the spacetime connection $\bar{\nabla}$. The second fundamental form, say B, of M is the symmetric bilinear form and is related with the Weingarten map by

$$B(X, Y) = g(\mathcal{W}_\xi X, Y) = g(\bar{\nabla}_X \xi, Y) \tag{5.2}$$

Using $\pounds_\xi g(X,Y) = g(\bar{\nabla}_X \xi, Y) + g(\bar{\nabla}_Y \xi, X)$ and $B(X,Y)$ symmetric in (5.2), we obtain

$$B(X, Y) = \frac{1}{2} \pounds_\xi g(X, Y), \quad \forall X, Y \in TM, \tag{5.3}$$

which is well-defined up to conformal rescaling (related to the choice of ξ). $B(X, \xi) = 0$ for any null normal ξ and for any $X \in TM$ implies that B has the same ξ degeneracy as that of the induced metric g.

Consider a class of lightlike hypersurfaces such that its second fundamental form B is conformally equivalent to its degenerate metric g. Geometrically, this means that (M, g) is totally umbilical in $\bar{M}$ if and only if there is a smooth function σ on M such that

$$B(X, Y) = \sigma g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{5.4}$$

It is obvious that above definition does not depend on particular choice of ξ. The name "umbilical" means that extrinsic curvature is proportional to the metric g. M is proper totally umbilical in $\bar{M}$ if and only if σ is non-zero on M. In particular, M is totally geodesic if and only if B vanishes, i.e., if and only if σ vanishes on M. It follows from the equations (5.3) and (5.4) that

$$\pounds_\xi g = 2\sigma g \quad \text{on} \quad M. \tag{5.5}$$

Thus, ξ is a conformal Killing vector (CKV) field in a totally umbilical M, with conformal function 2σ, which is Killing if and only if M is totally geodesic.

Now we need the following general result on totally umbilical submanifolds:

Proposition 24 Perlick [66] *Let (M, g) be a totally umbilical submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then,*

(a) a null geodesic of $\bar{M}$ that starts tangential to M remains within M (for some parameter interval around the starting point).

(b) M is totally geodesic if and only if every geodesic of $\bar{M}$ that starts tangential to M remains with in M (for some parameter interval around the starting point).

Considerable work has been done to show that (under certain conditions) totally geodesic lightlike hypersurfaces are black hole event (for example the Kerr family) or isolated horizons (see details with examples in [65], which include Killing horizons [62] as a special case). A Killing horizon is defined as the union $M = \bigcup M_s$, where M_s is a connected component of the set of points forming a family of lightlike hypersurfaces M_s whose null geodesic (as per above proposition) generators coincide with the Killing trajectories of nowhere vanishing ξ_s. The isolated horizon (IH) of a stationary asymptotically flat black hole is represented by the Killing horizon if M is analytic and the mixed energy condition holds for the stress-energy tensor of the Einstein field equations (see Section 3). For example, the following physical model of a spacetime can have a Killing horizon:

Physical model. Consider a 4-dimensional stationary spacetime $(\bar{M}, \bar{g})$ which is chronological, that is, $\bar{M}$ admits no closed timelike curves. It is well known [39] that a stationary $\bar{M}$ admits a smooth 1-parameter group, say G, of isometries whose orbits are timelike curves in $\bar{M}$. Denote by M' the Hausdorff and paracompact 3-dimensional Riemannian orbit space of the action G. The projection $\pi : \bar{M} \to M'$ is a principal R-bundle, with the timelike fiber G. Let $T = \partial_t$ be the non-vanishing timelike Killing vector field, where t is a global time coordinate function on M'. Then, the metric $\bar{g}$ induces a Riemannian metric $g_{{}_M}'$ on M' such that

$$\bar{M} = R \times M', \quad \bar{g} = -\,u^2\,(dt + \eta)^2 + \pi^\star\, g_{{}_M}',$$

where η is a connection 1-form for the R-bundle π and

$$u^2 = -g(T, T) > 0.$$

It is known that a stationary spacetime $(\bar{M}, \bar{g})$ uniquely determines the orbit data $(M',\, g_{{}_M}',\, u,\, \eta)$ as described above, and conversely. Suppose the orbit space M' has a non-empty metric boundary $\partial M' \neq \emptyset$. Consider the maximal solution data in the sense that it is not extendible to a larger domain $(\mathcal{M}',\, g'_{{}_{\mathcal{M}'}},\, u',\, \eta') \supset (M',\, g_{{}_M}',\, u,\, \eta)$ with $u' > 0$ on an extended spacetime $\mathcal{M}$. Under these conditions, it is known [39] that in any neighborhood of a point $x \in \partial M'$, either the connection 1-form η degenerates, or $u \to 0$. The second case implies that the timelike Killing vector T becomes null and M' degenerates into a lightlike hypersurface, say (M, g) of $\bar{M}$. Moreover, $lim(T)_{u \to 0} = V \in \mathcal{X}(TM)$ is a global null Killing vector field of M.

In the following we quote a result on physical interpretation of an ADM spacetime (see Section 3.2) which can admit a Killing horizon.

Theorem 25 [69] *Let $(\bar{M}, \bar{g})$ be an ADM spacetime evolved through a 1-parameter family of spacelike hypersurfaces Σ_t such that the evolution vector field is a null CKV field ξ on $\bar{M}$. Then, ξ reduces to a Killing vector field if and only if the part of ξ tangential to Σ_t is asymptotic everywhere on Σ_t for all t. Moreover, ξ is a geodesic vector field.*

There has been extensive study on black hole time independent Killing horizons for those spacetimes which admit a global Killing vector field. However, in reality, since the black holes are surrounded by a local mass distribution and expand by the inflow of galactic derbies as well as electromagnetic and gravitational radiation, their physical properties can best be represented by time-dependent black hole horizons. Thus, a Killing horizon (and for the same reason an isolated horizon) is not a realistic model. Since the causal structure is invariant under a conformal transformation, there has been interest in the study of the effect of conformal transformations on properties of black holes (see [67,68,70–72]). Directly related to the subject matter of this paper, we review the following work of Sultana and Dyer [70,71]:

Consider a spacetime $(\bar{M}, \bar{g})$ which admits a timelike conformal Killing vector (CKV) field. Let (M, g) be a lightlike hypersurface of $\bar{M}$ such that its null geodesic trajectories coincide with conformal Killing trajectories of a null CKV field (instead of Killing trajectories of the Killing horizon). This happens when a spacetime $\bar{M}$ becomes null on a boundary as a null geodesic hypersurface. Such a horizon is called *conformal Killing horizon*(CKH), as defined by Sultana-Dyer [70,71]. Consider a spacetime $(\bar{M}, G)$ related to a black hole spacetime $(\bar{M}, \bar{g})$ admitting a Killing horizon M generated by the null geodesic Killing field, with the conformal factor in $G = \Omega^2 \bar{g}$, where Ω is a non-vanishing function on $\bar{M}$. Under this transformation, the Killing vector field is mapped to a conformal Killing field ξ provided $\xi^i \bar{\nabla}_i \Omega \neq 0$. Since the causal structure and null geodesics are invariant under a conformal transformation, M still remains a null hypersurface of $(\bar{M}, G)$. Moreover, as per Proposition 24, the null geodesic of $\bar{M}$ that starts tangential to M will remain within M. Also, its null geodesic generators coincide with the conformal Killing trajectories. Thus, M is a CKH in (M, G).

Theorem 26 Sultana-Dyer [70] *Let (M, G) be a spacetime related to an analytic black hole spacetime (M, g) admitting a Killing horizon Σ_0, such that the conformal factor in $G = \Omega^2 g$ goes to a constant at null infinity. Then the conformal Killing horizon Σ in (M, G) is globally equivalent to the event horizon, provided that the stress energy tensor satisfies the week energy condition.*

Above paper also contains the case as to what happens when the conformal stationary limit hypersurface does not coincide with the CKH. For this case, they have proved a generalization of the weak rigidity theorem which establishes the conformal Killing property of the event horizon and the rigidity of its CKH.

Also, in [71] they have given an example of a dynamical cosmological black hole spacetime which describes an expanding black hole in the asymptotic background of the Einstein-de Sitter universe. The metric of such a spacetime is obtained by applying a time-dependent conformal transformation on the Schwarzschild metric, such that the result is an exact solution with the matter content described by a perfect fluid and the other a null fluid. They have also studied several physical quantities related to black holes.

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